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REAL OPTIONS AND PRODUCT LIFE CYCLES*

ABSTRACT

In this paper, I develop an option valuation framework that explicitly incorporates a product life cycle. I then use the framework to value the real option to change a project's capacity. Standard techniques for valuing real options typically ignore product life cycle models and specify instead a constant expected growth rate for demand or price. I show that this specification can lead to significant error in the valuation of capacity options. In particular, the standard technique tends to undervalue the option to contract capacity and overvalue the option to expand capacity. This result has important implications for capital investment decisions, especially in high-technology industries that feature regular introductions of newly improved products.

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Flexibility in the operation of a production facility is intuitively desirable, allowing managers to adapt to changing market conditions. When demand for a good falls, for example, perhaps due to the introduction of a substitute, the ability to reduce capacity and cut overhead costs would offset declining sales. Conversely, if demand for an innovative product exceeds all expectations, the ability to efficiently expand capacity would allow managers to swiftly capture market share. The ability to change capacity is one example of a real managerial option that can be valued using financial engineering. Real options may increase project value by allowing managers to direct more efficient production; however, the flexibility may increase the initial required investment in the project. Optimal managerial decision-making, therefore, requires a valuation framework for real options.

A variety of managerial options have been studied using techniques adapted from the valuation of financial options. Majd and Pindyck (1987), McDonald and Siegel (1986) and Pindyck (1988) value the option to defer production. Majd and Myers (1990) value the option to abandon a project early. McDonald and Siegel (1985) and Brennan and Schwartz (1985) value the option to temporarily shut down production. Triantis and Hodder (1990) value the option to change a project's output mix, while Kulatilaka (1993) values the option to change from one input good to another. Though previous studies differ in the options they value and the way they pose and solve the valuation problem, they generally share the assumption that changes in some underlying stochastic variable are governed by a single geometric diffusion, thereby permitting analytic solutions.

A number of important product markets, including high-technology goods such as semiconductors and pharmaceuticals, are characterized by well-defined product life cycles, indicating that simple stochastic processes may not always be appropriate. To illustrate, figure 1 depicts a clear bell-shaped pattern of sales for three successive generations of dynamic random access memory chips. As argued in Pisano and Wheelwright (1995), product life cycles such as these pose significant managerial challenges. The underlying stochastic process undergoes fundamental changes over the course of a product life cycle, naturally affecting expectations of a project's future profitability, and hence the value of a project's real options.

In this paper, I develop a real option valuation framework that explicitly incorporates a stochastic product life cycle. I represent the product life cycle using a regime-switching process. The cycle begins in a growth regime, characterized by increasing demand, and switches stochastically to a decay regime, in which demand generally falls. I value the option to change a project's capacity, and show that option values consistent with a product life cycle are significantly different than those from a standard model that makes simplifying assumptions about the demand process. This result has important managerial implications, since it implies that existing valuation techniques can lead to erroneous capital investment decisions.

The rest of this paper is organized as follows. Section 1 presents a product life cycle model and discusses the option valuation problem. Section 2 describes a valuation technique that explicitly incorporates a product life cycle. Section 3 contains a numerical example that illustrates the importance of correctly specifying demand dynamics. Section 4 concludes.

1. The Model

Suppose the manager of a firm is planning a production facility that will produce one output. The manager seeks to maximize the value of the project by choosing an optimal capacity. In addition, the manager seeks to value the option to change the facility's capacity in the future.

1.1. Fixed capacity

To begin, suppose the manager is determining the optimal capacity for a fixed capacity facility. In order to determine optimal capacity, the manager must specify a demand schedule for the project's output. In the real options literature, aggregate demand for a good at time *t*, Q_t^D, is typically assumed to be a function of price, P_t, as described by the following linear, stochastic schedule

$$Q_t^D = \theta_t - \lambda P_t \tag{1}$$

where θ is a stochastic demand parameter.[1] This specification results in a demand curve that randomly shifts towards and away from the origin, but always with the same slope.

The manager must also specify production costs in order to determine optimal capacity. A standard assumption sets cost equal to a quadratic function of quantity produced, Q_t^S, as follows:[2]

$$C\left(Q_t^S\right) = c_1 Q_t^S + \frac{c_2}{2}\left(Q_t^S\right)^2 . \tag{2}$$

The c_2 term results in a marginal cost that increases with quantity produced. For a fixed capacity plant, the c_2 term is consistent with the notion that costs increase as production quantity approaches the facility's capacity. For a flexible capacity plant, the c_2 term should be a decreasing function of capacity. I specify the following cost structure to reflect the impact of capacity, *M*, on production costs:

$$C\left(Q_t^S, M\right) = c_1 Q_t^S + \frac{c_2}{2M}\left(Q_t^S\right)^2 + c_3 M \tag{3}$$

where Q_t^S can never exceed M. Note that M now appears in the denominator of the c_2 term, so that an increase in capacity means that, for a given quantity produced, the facility is operating farther from capacity, and hence the costs of operating near capacity are reduced. In addition, I include the c_3 term to represent overhead costs. Overhead costs provide an incentive for managers to contract operations when excess capacity will likely never be used and to delay expansion until demand warrants extra capacity.

In addition to specifying demand and cost schedules, the manager must also consider the firm's competitive environment. I assume the following:

1 *The firm is a value-maximizing monopolist*

2 *Price adjusts instantly to equate quantities produced and demanded.*

These two assumptions permit a straightforward profit maximization problem with production quantity as the choice variable, as shown below. This analysis can be extended to consider equilibrium in an imperfectly competitive setting with a finite number of firms, but I focus on the monopoly case for expositional simplicity.[3]

Given the demand schedule in equation (1) and the cost structure in equation (3), the manager seeks to maximize the net present value (NPV) of the production facility by selecting an optimal capacity. Again suppose that capacity is fixed, so that M is constant over the life of the project. In the discrete-time framework of this paper, at the end of each period, a new value of θ is revealed, output is instantaneously produced in the quantity that maximizes current period profits, then output is sold at the price given by equation (1). Since quantity produced equals quantity demanded, the superscripts on Q_t are no longer necessary. To maximize current period profits, the manager must solve

$$\max_{Q_t}[\pi] = P(Q_t)Q_t - C(Q_t, M). \tag{4}$$

The manager's problem can be restated using the definitions in equations (1) and (3) as follows

$$\max_{Q_t}[\pi] = Q_t\left(\frac{\theta_t}{\lambda} - c_1\right) - Q_t^2\left(\frac{1}{\lambda} + \frac{c_2}{2M}\right) - c_3 M. \tag{5}$$

Setting the first derivative of the profit function with respect to Q_t equal to zero yields the following optimum:

$$Q_t^* = \frac{\theta_t - \lambda c_1}{2 + \lambda c_2 / M}. \tag{6}$$

Since quadratic production costs ensure that the second derivative of the profit function is negative, this optimum maximizes profits. The facility's capacity restricts production and of course production can never fall below zero. These bounds relate actual production, Q_t, to the optimum as follows:

$$Q_t = \max\left(0, \min\left(Q_t^*, M\right)\right). \tag{7}$$

Define the profit function $\pi_t^*(\theta_t, M)$ as the period *t* profit given the optimal production consistent with the bounds in equation (7). Assuming a finite life of *T* years, the project's NPV is given by

$$NPV(M) = -c_4 M + \sum_{t=1}^{T} e^{-rt} E\left[\pi_t^*(\theta_t, M)\right] \tag{8}$$

where c_4 is the cost of installing one unit of capacity.[4] The manager's problem is to choose the capacity that maximizes this function. An interesting issue is selecting the appropriate project life *T*. If demand continues to grow unexpectedly over time, presumably the project life should be extended. If demand shrinks quickly, the manager may want to cut short the project life. This highlights the need to value the option to change capacity when planning the production facility. There are at least two ways to deal with the issue of project life. One approach is to make assumptions about the terminal value of the project at some date T. Alternatively, one could try successively larger values for T until project value is unchanged. This issue will be discussed further in section 3.

The NPV of the facility depends on the capacity, M, since capacity is costly to install, generates overhead costs, and places an upper bound on annual production. Given a capacity level, the stochastic demand parameter θ drives the NPV. Different assumptions about the stochastic process that governs the evolution of θ naturally affect both the distribution of future values of θ and the expectations of conditional profits.

1.2. Flexible capacity

Suppose now that the manager of the firm seeks to value the option to switch at any time between a discrete number of capacity levels.[5] Pindyck (1988) studies capacity choice and expansion options. He motivates the value of deferring the installation of capacity by assuming that this investment is irreversible. However, as discussed in Baldwin and Clark (1997), modern design and production techniques focus on modularity. One implication of modular production is that the option to salvage excess capacity can be valuable, since capacity can be readily reallocated to different projects. In other words, investment in capacity is at least partially reversible. Majd and Myers (1990) recognize this and value the option to abandon a project for its salvage value. In this paper I consider both capacity expansion and contraction, nesting previous problems as special cases.

To model the cost of changing capacity, I define the function $S(M_1,M_2)$ to represent the one-time cash flow associated with changing capacity from M_1 to M_2:

$$S(M_1,M_2) = s_1 c_4 (M_2 - M_1) + s_3 \text{ when } M_2 > M_1$$
$$S(M_1,M_2) = s_2 c_4 (M_1 - M_2) + s_3 \text{ when } M_1 > M_2. \quad (9)$$

Both s_1 and s_2 can be positive or negative and represent a percentage of the initial installment cost of one unit of capacity. Increases in capacity generally require cash outflows so that s_1 is typically negative. Capacity decreases could generate cash outflows (clean up costs) or inflows (salvage value of machinery) so that s_2 could be positive or negative, although in this paper I consider only positive values. Fixed switching costs are captured by s_3. The cost function in equation (9) could easily be extended to allow for time-varying switching costs. These might be appropriate, for example, when the purchase price of additional capacity or the salvage value of excess capacity changes over time.

The value of capacity flexibility can be determined by computing the incremental increase in project NPV when capacity is allowed to change. Project NPV can be expressed symbolically as before, except now switching costs are included each period, and capacity is of course allowed to change each period:

$$NPV(M_0) = -c_4 M_0 + \sum_{t=1}^{T} e^{-rt} \left\{ E\left[\pi_t^* (\theta_t, M_{t-1}) + S(M_{t-1}, M_t) \right] \right\}. \quad (10)$$

Note that capacity at date $t-1$ places an upper bound on production at date t. Changes in capacity are assumed to take effect the following period, consistent with the notion that it takes some time for capacity to be changed.

Left out of the analysis thus far is the optimal capacity policy. At each date, the manager will choose a new capacity level that optimally addresses the tradeoff between the switching cost and the change in expected future profits. The change in expected future profits is complex, incorporating the impact of a new capacity level on cash flows, both through changing overhead costs and by capping future production, and the impact on future option values. The standard valuation approach is dynamic programming, as discussed further in section 2.

1.3. Demand dynamics

To proceed with the valuation problem outlined above, the manager must specify the evolution of the demand parameter θ in order to compute expected future profits. A standard assumption in the real options literature is that the underlying stochastic variable is governed by a geometric diffusion. Although a geometric diffusion is tractable, it can have some undesirable implications. When the growth rate in demand is governed by a geometric diffusion, expected demand grows at a constant rate in perpetuity, which seems unrealistic. The simple specification ignores the possibility that the dynamics of demand for the underlying good will change in a fundamental way over the course of the project's life. Product life cycle models, in contrast, are based on the notion that demand decays at some point, due to market saturation, introduction of competing goods, development of superior technology, or changing tastes. The Bass (1969) diffusion model, for example, implies that sales exhibit exponential growth to some peak and then decay exponentially. Product life cycle models can be viewed as regime-switching models, in which different stages of the product's life are characterized by fundamentally different stochastic demand schedules. Further, the duration of each stage of the product life cycle is unknown.[6]

If a product life cycle does indeed capture the demand dynamics of the underlying good, then the simple specification of a single demand schedule can lead to significant valuation error. The assumption of constant growth intuitively undervalues the option to contract or abandon the project, since it underestimates the probability of low demand in the future. Likewise, the assumption of constant growth intuitively overvalues the option to expand the project, since it implies that demand is expected to increase forever. The complexity of the capacity options and the regime-dependent demand schedules of the product life cycle make formal proofs of these conjectures elusive. In section 3, however, I show that they hold true numerically over a large range of parameter values.

To model a product life cycle, I specify a linear demand schedule as before, in which the continuously compounded growth rate of the demand parameter θ is normally distributed. Now, however, the parameters of the normal distribution differ across a growth regime and a decay regime.[7] The drift in the growth regime, μ_g, is assumed to be positive, whereas the drift in the decay regime, μ_d, is assumed to be negative. I assume that the slope of the demand curve is equal across regimes, although this assumption is easily relaxed. This specification results in a demand curve that randomly shifts towards and away from the origin, as before, but in the growth regime it is expected to shift away from the origin and in the decay regime it is expected to shift towards the origin. I also assume that demand begins in the growth regime and switches at most once to the decay regime over the course of the project's life. This assumption is also easily relaxed, but is appropriate for a product life cycle.

I assume that the manager always knows the current regime.[8] This implies that the manager knows the distribution that will govern the next observation of demand, but not future distributions. For some products, knowledge of the current regime may be an unrealistic assumption. However, for important cases such as semiconductors and pharmaceuticals, regimes are clearly identified by the introduction of competing or next-generation products.

The probability distribution governing the switch from growth to decay in the product life cycle is critical to project NPV, since the time spent in each regime affects expected demand and period profits. I set the probability of switching from growth to decay over the next year equal to a cumulative normal distribution function of the time elapsed since the beginning of the project. The time-dependency is motivated by the regular introductions of new generations of products that replace existing standards, as in the semiconductor industry. Suppose, for example, that the mean of the switching distribution is five years and the standard deviation is one year. This switching distribution implies that the probability of switching, conditional on still being in the growth regime, is about 2.28% in the fourth year, 15.87% in the fifth year, 50% in the sixth year, 84.13% in the seventh year, 97.72% in the eighth year, and essentially 100% thereafter.

Two alternatives to the time-dependent switching distribution are to set the probability of switching equal to some function of the demand parameter θ, as in Norton and Bass (1987), or to set the probability of switching to some function of cumulative sales, consistent with the pure diffusion model in Bass (1969). These alternative switching distributions are motivated by the notion that the more popular a product, the more likely is the introduction of a competing or substitute product which erodes sales of the original. The valuation technique outlined in the next section can easily handle the first alternative, but the second introduces a path-dependency to the decision-making process that would make the problem significantly more difficult.

The cost structure in the product life cycle is again given by equation (3), and parameters are assumed equal across regimes. This means that optimal supply, given a capacity level and the current level of demand is again given by equation (6). A cost structure that does not change over time or cumulative production implies that there are no learning curve effects, which can be important in a real options context as noted by Majd and Pindyck (1989). The numerical technique described in the next section could be extended to allow for costs that decline over time or over stages in the product life cycle.

2. Project and Option Valuation

Real options such as capacity flexibility are complicated since they can be viewed as a portfolio of nested options with a continuum of exercise dates. The options are nested because the manager can choose to change capacity at any date, and changing capacity at any date affects the value of all future options. This complexity, along with the uncertainty regarding future demand regimes, requires the use of numerical methods to solve for project and option value.

2.1. Dynamic programming

The standard numerical approach for valuing real options is dynamic programming. The basic idea is to establish a discrete-valued lattice of possible future values of the underlying stochastic variable θ and the choice variable M. Project value is computed at each node in the lattice, conditional on the level of demand and capacity. To allow for early option exercise, the valuation procedure begins at the end of project life and folds back recursively to the present. For terminal nodes, project value is simply the final cash flow. For intermediate nodes, project value is the sum of the current period's cash flow and the discounted, expected project value in the next time period. The stochastic process governing the evolution of the stochastic demand parameter determines how the expectations are formed. The current value of the project is calculated at the seed node, and option values are computed as the difference between project value with and without the managerial flexibility of interest.

To proceed, we need to map the model established in section 1 to this lattice. Recall that the project has some finite life of T years. The lattice is constructed with some integer number, n, time steps, so that t, which equals T/n, is the duration of each step in years. As the time step shrinks, the quantity demanded and production capacity M are scaled by t. Suppose the annual continuous rate of change of the demand parameter θ is assumed normally distributed with a mean μ and volatility σ. The continuous rate of change in the demand parameter θ over the time step t is therefore normally distributed with a mean of μt and volatility $\sigma\sqrt{t}$.

One critical feature of the valuation procedure is determining an appropriate discount rate. There are two sources of risk in the model. The first is the uncertainty regarding the switch from growth to decay. I assume that this risk can be diversified away by investment in other projects. This makes sense for firms that are constantly replacing existing products with improved ones. Since the product life cycle risk is diversifiable, it is not priced in the market. The second source of risk is demand for the product, since demand is stochastic within each stage of the product life cycle. I assume that traded assets span changes in demand and employ risk-neutral valuation, in which the project's cash flows are discounted at the riskless rate of interest and growth rates in demand are adjusted to reflect a risk premium.[9] The size of the risk-adjustment depends on the market price of demand risk. While risk-adjustments are typically used in an investments context, they are appropriate for all types of underlying variables. A risk-adjustment simply represents the trade-off between risk and return that is made in the marketplace for securities or projects that are dependent on some underlying variable, regardless of whether the underlying variable is the return on a traded asset, the temperature, or aggregate demand for some product. There are several ways to estimate the market price of risk. One way is to estimate it from a time-series of security prices using an equilibrium pricing model such as the Capital Asset Pricing Model, as in Constantinides (1978). Another way is to infer the risk-adjusted growth rates directly from the prices of derivative securities, as described in Hull (1997), pages 296-298.

One other feature of the product life cycle model requires special treatment. Recall that the probability of switching from growth to decay at any time over the course of the next year is given by a cumulative normal distribution function of elapsed time. We need to compute the probability of switching regimes over the next time step. One approach is to assume that the probability of switching is constant over the course of the year. Suppose there are *m* time steps per year. The probability of switching from growth to decay over the next time step is set so that if it were constant over *m* time steps, then the cumulative probability of switching over the course of the year is equal to the original switching probability. Let P denote the probability of switching over the course of the coming year. We need to choose *p*, the probability of switching over the next time step, so that

$$1-(1-p)^{m}=P. \tag{11}$$

Equation (11) implies that

$$p=1-(1-P)^{\frac{1}{m}}. \tag{12}$$

2.2. Lattice topology

A standard method for achieving computational efficiency in a dynamic program is to approximate normal distributions by binomial distributions, as outlined in Cox, Ross, and Rubinstein (1979). Each node in a binomial lattice has two free parameters. These can be expressed as the probability of traveling along the upper branch, denoted by π, and the continuously compounded rate of return of the variable after traveling along the upper branch, denoted by ϕ. These parameters are chosen so that the first two moments of the variable implied by the lattice match the first two moments implied by the underlying risk-neutral distribution.

A single binomial lattice can be used to represent the possible future paths of demand when changes in demand are distributed normally. In the product life cycle model of this paper, however, there are two relevant distributions, one for each regime. Since a binomial lattice can represent either distribution, four branches are sufficient to represent two arbitrary normal distributions simultaneously. In order to represent two distributions efficiently, however, I construct a lattice with five branches at each node, as in Bollen (1998). The reason for the efficiency gain is that in a four-branch lattice the branches do not generally recombine efficiently, so that the number of nodes at time step t is t^2. The additional flexibility afforded by the fifth branch means that the branches can be spaced evenly while still maintaining a match between the distributions implied by the lattice and the risk-neutral distributions specified for the underlying variable. See figure 2 for an illustration. The five evenly spaced branches result in a lattice that has $4t - 3$ nodes at time step t. In a 500-step tree, for example, the four-branch lattice has a total of 41,791,750 nodes whereas the five-branch lattice has only 499,500, a reduction of about 99%.

The lattice is constructed by representing one stage of the product life cycle by a binomial and the other stage by a trinomial. Bollen (1998) explains how to decide which regime to represent by the trinomial. The step size of the trinomial lattice can be adjusted to space the five branches evenly. This technique can be extended to represent additional regimes in a product life cycle. Suppose, for example, that a third regime represents an intermediate stage in the product life cycle. One could use a decision rule analogous to the one for the two-regime case to decide which two of the three regimes to represent by a trinomial and which to represent by a binomial. The step sizes of the trinomials could then be adjusted to space the branches evenly. This would result in seven recombining branches at each node.

2.3. Computing project and option values

With the lattice topology established we can now value the project and related options. Calculations begin at the end of the lattice and work backward to the present. Project value is computed at each node, conditional on the level of θ, the demand regime, and the capacity level of the prior node. Since the optimal capacity level of the prior node is unknown, all possible prior capacity levels are considered. This is why I must assume that only a discrete number of capacity levels are possible.

For terminal nodes, project value is equal to the last period's cash flow given the terminal level of the demand parameter θ and the capacity level of the prior time step. Since period profits are a function only of θ and capacity, project value is equal across demand regimes at the terminal nodes. For intermediate nodes, project value is equal to the sum of the current period's cash flow and the expected, discounted future project value assuming an optimal production strategy. Capacity is allowed to change at each node and any changes are assumed to take effect in the following node.[10] To model capacity choice, given each capacity level of the prior node, value is maximized by searching over switches to all other possible capacity levels. Moreover, the expected, discounted future project value at a candidate capacity level incorporates the possibility of switching from growth to decay at some point over the next time interval.

To illustrate, let $NPV(i,j,k,t)$ indicate project value conditioned on being in regime j, where i indicates the level of θ with i = 1 the highest level at each time step, k indicates the capacity level of the prior node, and t the time step. Let $EV(i,j,K,t)$ represent the expected, discounted future project value given a switch to the candidate capacity level K. For each level of capacity k, NPV is maximized by searching over the switches to all other possible capacity levels:

$$NPV(i,j,k,t) = \max_{K}\left[\pi^{*}(i,k) + S(k,K) + EV(i,j,K,t)\right] \tag{19}$$

In the pentanomial lattice, the form of *EV* differs across the two regimes since each regime is represented by a different set of branches. Suppose, for example, that the decay regime has the smaller step size and is represented by the trinomial. At time step *t*, the expected, discounted future project value conditional on still being in the growth regime at time *t*, with demand level *i* and candidate capacity level *K* is given by

$$EV(i,g,K,t)=e^{-rt}\begin{bmatrix}(1-p(t))\left(\pi_{g,u}NPV(i,g,K,t+1)+\pi_{g,d}NPV(i+4,g,K,t+1)\right)\\+\,p(t)\begin{pmatrix}\pi_{d,u}NPV(i+1,d,K,t+1)+\pi_{d,m}NPV(i+2,d,K,t+1)+\\ \pi_{d,d}NPV(i+3,d,K,t+1)\end{pmatrix}\end{bmatrix},\qquad(20)$$

where $p(t)$ is the probability of switching from growth to decay and π is a conditional branch probability. The first subscript on π denotes the regime and the second subscript denotes the up, middle, or down branch. See figure 2 for an illustration. The branch probabilities are computed from the risk-adjusted mean and volatility of each regime as explained in Bollen (1998). Conditional on staying in the growth regime, the demand parameter travels along the outer two branches, consistent with the first line in the brackets of equation (20). Conditional on switching to the decay regime, the demand parameter travels along the inner three branches. If demand has switched to the decay regime by time *t*, the expected, discounted future project value is

$$EV(i,d,K,t)=e^{-rt}\begin{bmatrix}\pi_{d,u}NPV(i+1,d,K,t+1)+\pi_{d,m}NPV(i+2,d,K,t+1)+\\ \pi_{d,d}NPV(i+3,d,K,t+1)\end{bmatrix}.\qquad(21)$$

Recall that the switching probability is assumed to be a cumulative normal distribution function of elapsed time. It should be clear now that this transition probability can also easily be some function of θ, since the computations in equations (20) and (21) are conditional on θ. Note also that a switch can occur only from growth to decay, so only when j = g does the possibility of a switch enter the computation. However, the approach can easily be generalized to allow for switches back and forth between regimes.

The induction proceeds backward to the present. The initial node gives the current value of the project including any capacity options for each initial capacity level. The initial capacity level that maximizes the project's NPV is optimal.

3. Numerical Example

This section presents a numerical analysis of a project that produces a single output. Demand is governed by a stochastic product life cycle. The project value and the values of several capacity options are computed under two sets of assumptions. First, the product life cycle is explicitly incorporated in the analysis using the methodology outlined in section 2. Second, the analysis is duplicated under the alternative assumption of geometric Brownian motion (a single regime process) using a simple binomial lattice. Differences between the two sets of results measure the potential for specification error when the product life cycle is ignored. Since this is a numerical example, the results can not always be generalized; nevertheless some insight can be gained from examining the sensitivity of project and option values to parametric assumptions.

The single demand process must be specified carefully in order to meaningfully compare the two analyses. I select the parameters of the geometric Brownian motion to reflect the entire product life cycle by combining the growth and decay regimes. The variance used for the single demand process is a weighted average of the two regimes' variances, where the expected fractions of the project's life spent in each regime serve as the weights. The growth rate for the single demand process is set to generate a project value with fixed capacity equal to the project value consistent with the product life cycle. Since I set the project value of the single demand process equal to the project value of the product life cycle, differences in option values highlight the impact of different assumptions about demand.

In this numerical example, the default parameter values are as follows. The project has a 10 year life. The switch from growth to decay in the product life cycle occurs with probability equal to the cumulative normal distribution function of elapsed time, with a mean of 5 years and a standard deviation of 1 year. Project value is maximized by searching over a range of capacities from 0.0 to 5.0 units per year, in increments of .05 units. Changes in demand are distributed normally. For the growth phase of the product life cycle, the annual risk-adjusted mean is 30% and the volatility is given in the vertical axes of the panels in the table. For the decay phase, the annual risk-adjusted mean is –30% with volatility 20%. Initial demand is 1.0 and the demand curves have slopes of 1.0 in both phases of the product life cycle. The production cost parameters are: $c_1 = .10$, $c_2 = .50$, c_3 is given in the horizontal axes of the panels in the table, and $c_4 = 2.0$. The switching cost parameters are: $s_1 = -1.0$, $s_2 = .85$, and $s_3 = .05$. The riskless rate of interest is 10%.

Table 1 shows project and option values and corresponding optimal initial capacity levels. Panels A and B show project value with no options. Project value is decreasing in overhead cost; for a given capacity level, added overhead cost reduces profitability. Note that an increase in overhead cost results in a smaller optimal capacity, so that production quantity and revenue are also limited. Project value is increasing in growth volatility. The larger is volatility, the greater is the probability that demand will grow significantly more or significantly less than anticipated. For the parameter values chosen here, the upside associated with unusually high growth rates outweighs the downside associated with unusually low growth rates, so that project value is increasing in growth volatility.

Panels C and D show the value of the contraction option, which is computed as the difference between project value when capacity can be reduced from its initial level and project value with fixed capacity. A capacity level of zero is equivalent to abandoning the project since in this scenario capacity cannot be added. Panels E and F show the value of the expansion option, which is computed as the difference between project value when capacity can be increased from its initial level and project value with fixed capacity. Here a capacity level of zero is equivalent to delaying the project. Both contraction and expansion options are more valuable the higher is volatility, consistent with the standard relation between option value and volatility. Also, as noted in Pindcyk (1988), the value of expansion options are quite significant relative to project value.

More important, panels C through F show that the single demand process undervalues the contraction option and overvalues the expansion option, as conjectured earlier in the paper. For example, when $c_3 = .50$ and $\sigma_g = 10\%$, the single regime's contraction option is worth about 50% less than the product life cycle's contraction option, whereas the single regime's expansion option is worth about 150% more than the product life cycle model's expansion option. Panels G and H show the value to both increase and decrease capacity. Regardless of the underlying demand specification, the value of capacity flexibility is quite significant. For both specifications, for example, when $c_3 = .50$ and $\sigma_g = 30\%$, the general option value is worth about double the fixed capacity project itself!

Table 1 also shows the impact capacity flexibility can have on the optimal initial capacity of a production facility. In panels A and B, we see that initial capacity decreases with increases in overhead costs. An increase in overhead provides an obvious incentive to pare capacity that will likely never be used. Further, we see that initial capacity increases with growth volatility.

The increased capacity is more able to capitalize on future high demand. In panels C and D, we see that when managers have the option to contract, initial optimal capacity is higher than when capacity is fixed. And in panels E and F, we see that when managers have the option to expand, initial optimal capacity is significantly lower. Pindyck (1988) also found this result. Lower initial capacity reduces overhead costs until capacity is warranted.

Table 2 shows the percentage change in the project and option values when the project life is extended to 11 years. All other parameters are the default values. For the product life cycle model, project and option values are relatively unchanged when project life is extended to 11 years, indicating that the assumed 10 year life of table 1 is appropriate. The reason is that, given a mean switch point of five years, we expect the project will have been in the decay phase for a number of years by the end of year 10. This means that demand is at a low level, and is expected to drop more in year 11. When overhead costs are high, the project may in fact be operating at a loss by year 10; hence the contraction option is increasing in overhead costs whereas the project NPV and expansion option are decreasing in overhead costs.

Note, though, that the single regime project and option values are all much higher for an 11 year life. The reason is that when we expect demand to grow in perpetuity, it is expected to be quite high by the end of year 10, and hence even higher in year 11. Project and expansion options are therefore much more valuable in year 11. Contraction options have also increased in value. When demand is at a high level, the probability of a significant decline due to volatility is also high, so that in year 11 there is a substantial probability that the manager may wish to reduce capacity. In fact, project and option values will continue to increase when the project life is extended so long as NPV grows faster than the riskless rate of interest. This illustrates how the unrealistic assumption of a single, constant expected growth rate of demand can be problematic.

The product life cycle model, in contrast, provides a natural and meaningful motivation for a finite project life.

Table 3 shows the sensitivity of project and option values to the parameters of the switching distribution that governs the evolution of the product life cycle. All other parameter values are the default values with $\sigma_g = 20\%$. Panel A shows that project value is higher the higher is the mean of the switching distribution, since this implies that the life cycle will likely remain in a state of growth for a longer period of time. Panel A also shows that project value is decreasing in the uncertainty surrounding the switch date. For the production parameters chosen, the upside associated with staying in the growth regime longer than expected is outweighed by the downside when the switch to decay occurs sooner than expected. Panels B and C show that both options are much more valuable the higher is the mean of the switching distribution. The contraction option becomes more valuable because capacity levels are greater when demand is expected to reach a higher level; as a result, the overhead savings from cutting capacity in the decay regime are greater. The expansion option becomes more valuable because as demand is expected to reach a higher level, expected overhead savings from delaying capacity are greater.

Table 4 lists the sensitivity of real option values to production modularity, as measured by the cost of either contracting or expanding capacity. The parameters that vary in the panels are overhead (c_3), unit expansion cost (s_1), and unit salvage value (s_2). All other parameter values are the default values with $\sigma_g = 20\%$. Panel A shows that contraction option value is increasing in salvage value. The higher the salvage value, the greater the payoff to exercising a capacity option per unit of capacity. Also, the higher the salvage value, the greater the initial capacity, so the expected overall payoff to cutting capacity is increased. The relation between contraction option value and overhead cost is more complex. There are two opposing effects at work. An increase in overhead costs increases the cost savings per unit of capacity when capacity is reduced. However, an increase in overhead costs decreases the initial capacity, so the expectation of future excess capacity, and hence the expected payoff to exercising the contraction option, is lower.

Panel B of table 4 shows that the expansion option value is decreasing in per unit expansion cost. Since an increase in expansion cost increases the price of adding capacity, it lowers the value of the expansion options. Expansion option value is increasing in overhead cost for two reasons. First, the payoff to delaying installation of each unit of capacity is greater. Second, initial capacity is lower, so that the total payoff to delaying installation of capacity is greater.

In summary, this analysis shows that when demand is governed by a product life cycle, real options can be valued with great error when a single demand process is specified instead. This result indicates that the option valuation method outlined in this paper can be of significant practical use. Further, option values are quite sensitive to the efficiency with which capacity is changed. This result can be used to justify investment in production modularity, so that blocks of capacity can be added to existing capacity cheaply, and excess capacity can be salvaged for other uses.

4. Conclusion

The product life cycle characterizes the demand dynamics of many consumer industries. In this paper, I present a valuation framework for real options when demand is governed by a stochastic product life cycle. I then show that a standard method, which ignores the product life cycle, can undervalue the contraction option, since it underestimates the probability that demand will most likely fall at some point in the future, and overvalue the expansion option, since it implies that demand is expected to grow indefinitely. This result has important implications for capital investment decisions, and highlights the need for flexible valuation techniques.

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Table 1

Project and Option Value

Listed are project values and the values of several capacity options in bold and corresponding optimal initial capacities in italics. The project and options are valued under the assumption that demand is governed by a product life cycle (PLC) and a geometric Brownian motion (GBM).

Panel A: PLC Project Value

	Overhead Cost				
σ_g	*0.10*	*0.20*	*0.30*	*0.40*	*0.50*
0.10	**6.15**	**5.34**	**4.62**	**3.98**	**3.41**
	1.35	*1.20*	*1.10*	*0.95*	*0.85*
0.20	**6.61**	**5.68**	**4.87**	**4.16**	**3.54**
	1.60	*1.35*	*1.20*	*1.05*	*0.95*
0.30	**7.46**	**6.33**	**5.37**	**4.54**	**3.82**
	1.95	*1.65*	*1.40*	*1.20*	*1.05*

Panel B: GBM Project Value

	Overhead Cost				
σ_g	*0.10*	*0.20*	*0.30*	*0.40*	*0.50*
0.10	**6.15**	**5.34**	**4.62**	**3.98**	**3.41**
	1.50	*1.30*	*1.15*	*1.00*	*0.90*
0.20	**6.61**	**5.68**	**4.87**	**4.16**	**3.54**
	1.70	*1.45*	*1.25*	*1.10*	*0.95*
0.30	**7.46**	**6.33**	**5.37**	**4.54**	**3.82**
	2.05	*1.70*	*1.45*	*1.25*	*1.05*

Panel C: PLC Contraction Option Value

	Overhead Cost				
σ_g	*0.10*	*0.20*	*0.30*	*0.40*	*0.50*
0.10	**1.16**	**1.11**	**1.06**	**1.01**	**0.96**
	1.75	*1.55*	*1.40*	*1.25*	*1.15*
0.20	**1.55**	**1.48**	**1.41**	**1.34**	**1.27**
	2.25	*2.00*	*1.75*	*1.55*	*1.40*
0.30	**2.38**	**2.30**	**2.21**	**2.12**	**2.02**
	3.30	*2.85*	*2.55*	*2.25*	*2.00*

Panel D: GBM Contraction Option Value

	Overhead Cost				
σ_g	*0.10*	*0.20*	*0.30*	*0.40*	*0.50*
0.10	**0.79**	**0.69**	**0.61**	**0.55**	**0.49**
	1.90	*1.60*	*1.40*	*1.25*	*1.10*
0.20	**1.01**	**0.89**	**0.80**	**0.72**	**0.65**
	2.25	*1.90*	*1.65*	*1.45*	*1.30*
0.30	**1.47**	**1.32**	**1.20**	**1.09**	**1.00**
	3.00	*2.50*	*2.15*	*1.90*	*1.65*

Panel E: PLC Expansion Option Value

	Overhead Cost				
σ_g	*0.10*	*0.20*	*0.30*	*0.40*	*0.50*
0.10	**0.98**	**1.12**	**1.22**	**1.28**	**1.32**
	0.45	*0.35*	*0.30*	*0.25*	*0.20*
0.20	**2.54**	**2.80**	**2.98**	**3.11**	**3.20**
	0.40	*0.35*	*0.30*	*0.25*	*0.20*
0.30	**4.31**	**4.76**	**5.09**	**5.34**	**5.51**
	0.35	*0.25*	*0.20*	*0.15*	*0.10*

Panel F: GBM Expansion Option Value

	Overhead Cost				
σ_g	*0.10*	*0.20*	*0.30*	*0.40*	*0.50*
0.10	**2.36**	**2.72**	**3.01**	**3.23**	**3.40**
	0.35	*0.30*	*0.25*	*0.20*	*0.15*
0.20	**3.34**	**3.78**	**4.13**	**4.40**	**4.60**
	0.35	*0.30*	*0.20*	*0.15*	*0.10*
0.30	**4.83**	**5.44**	**5.91**	**6.26**	**6.54**
	0.30	*0.25*	*0.20*	*0.15*	*0.10*

Panel G: PLC General Option Value

	Overhead Cost				
σ_g	*0.10*	*0.20*	*0.30*	*0.40*	*0.50*
0.10	**2.37**	**2.58**	**2.74**	**2.85**	**2.94**
	0.45	*0.35*	*0.30*	*0.25*	*0.20*
0.20	**3.93**	**4.26**	**4.51**	**4.70**	**4.83**
	0.40	*0.35*	*0.30*	*0.25*	*0.20*
0.30	**5.74**	**6.27**	**6.68**	**6.99**	**7.22**
	0.35	*0.30*	*0.25*	*0.20*	*0.15*

Panel H: GBM General Option Value

	Overhead Cost				
σ_g	*0.10*	*0.20*	*0.30*	*0.40*	*0.50*
0.10	**3.25**	**3.60**	**3.86**	**4.07**	**4.22**
	0.35	*0.30*	*0.25*	*0.20*	*0.15*
0.20	**4.21**	**4.64**	**4.97**	**5.22**	**5.40**
	0.35	*0.30*	*0.25*	*0.20*	*0.15*
0.30	**5.68**	**6.27**	**6.73**	**7.07**	**7.33**
	0.30	*0.25*	*0.20*	*0.15*	*0.10*

Table 2

Sensitivity to Project Life

Listed are the percentage change in project values and the values of several capacity options when project life is increased from 10 to 11 years. The project and options are valued under the assumption that demand is governed by a product life cycle (PLC) and a geometric Brownian motion (GBM).

Panel A: % Change in PLC Project Value

σ_g	Overhead Cost *0.10*	*0.20*	*0.30*	*0.40*	*0.50*
0.10	**1.41%**	**0.88%**	**0.33%**	**-0.24%**	**-0.88%**
0.20	**1.59%**	**1.05%**	**0.47%**	**-0.12%**	**-0.73%**
0.30	**1.80%**	**1.22%**	**0.64%**	**0.05%**	**-0.61%**

Panel B: % Change in GBM Project Value

σ_g	Overhead Cost *0.10*	*0.20*	*0.30*	*0.40*	*0.50*
0.10	**44.18%**	**45.83%**	**47.64%**	**49.76%**	**52.09%**
0.20	**45.86%**	**47.37%**	**49.09%**	**51.11%**	**53.42%**
0.30	**48.69%**	**50.04%**	**51.65%**	**53.54%**	**55.79%**

Panel C: % Change in PLC Contraction Option Value

σ_g	Overhead Cost *0.10*	*0.20*	*0.30*	*0.40*	*0.50*
0.10	**0.74%**	**3.28%**	**5.48%**	**7.28%**	**8.90%**
0.20	**1.62%**	**3.82%**	**5.72%**	**7.25%**	**8.51%**
0.30	**2.24%**	**4.07%**	**5.43%**	**6.49%**	**7.45%**

Panel D: % Change in GBM Contraction Option Value

σ_g	Overhead Cost *0.10*	*0.20*	*0.30*	*0.40*	*0.50*
0.10	**13.21%**	**14.22%**	**15.36%**	**16.32%**	**17.73%**
0.20	**20.32%**	**22.04%**	**23.58%**	**25.12%**	**26.57%**
0.30	**29.58%**	**31.71%**	**33.48%**	**35.05%**	**36.45%**

Panel E: % Change in PLC Expansion Option Value

σ_g	Overhead Cost *0.10*	*0.20*	*0.30*	*0.40*	*0.50*
0.10	**-0.04%**	**-1.01%**	**-1.96%**	**-2.94%**	**-3.82%**
0.20	**0.75%**	**0.44%**	**0.10%**	**-0.32%**	**-0.78%**
0.30	**1.06%**	**1.05%**	**0.88%**	**0.64%**	**0.40%**

Panel F: % Change in GBM Expansion Option Value

σ_g	Overhead Cost *0.10*	*0.20*	*0.30*	*0.40*	*0.50*
0.10	**48.02%**	**49.74%**	**51.25%**	**52.59%**	**53.93%**
0.20	**40.80%**	**43.29%**	**45.17%**	**46.78%**	**48.15%**
0.30	**31.44%**	**35.05%**	**37.60%**	**39.57%**	**41.17%**

Panel G: % Change in PLC General Option Value

σ_g	Overhead Cost *0.10*	*0.20*	*0.30*	*0.40*	*0.50*
0.10	**0.48%**	**1.55%**	**2.28%**	**2.76%**	**3.11%**
0.20	**0.88%**	**1.61%**	**2.12%**	**2.43%**	**2.62%**
0.30	**1.17%**	**1.80%**	**2.16%**	**2.38%**	**2.54%**

Panel H: % Change in GBM General Option Value

σ_g	Overhead Cost *0.10*	*0.20*	*0.30*	*0.40*	*0.50*
0.10	**35.05%**	**37.96%**	**40.28%**	**42.22%**	**43.99%**
0.20	**31.95%**	**35.07%**	**37.44%**	**39.41%**	**41.08%**
0.30	**26.18%**	**30.00%**	**32.75%**	**34.90%**	**36.65%**

Table 3

Sensitivity of Project and Option Value to Product Life Cycle

Listed are project values and the values of several capacity options in bold and corresponding optimal initial capacities in italics. The project and options are valued under the assumption that demand is governed by a product life cycle. The probability of switching from growth to decay is given by the cumulative normal distribution of elapsed time with mean as given in the vertical axes and volatility in the horizontal axes of the panels below.

Panel A: Project Value

Switch Mean	Switch Volatility *1.00*	*1.50*	*2.00*	*2.50*	*3.00*
3.00	**0.30**	**0.18**	**0.09**	**0.04**	**0.02**
	0.25	*0.20*	*0.15*	*0.10*	*0.05*
4.00	**1.31**	**0.97**	**0.65**	**0.42**	**0.26**
	0.50	*0.45*	*0.35*	*0.30*	*0.25*
5.00	**3.54**	**2.81**	**2.07**	**1.45**	**1.00**
	0.95	*0.80*	*0.70*	*0.60*	*0.45*

Panel B: PLC Contraction Option Value

Switch Mean	Switch Volatility *1.00*	*1.50*	*2.00*	*2.50*	*3.00*
3.00	**0.57**	**0.56**	**0.53**	**0.49**	**0.46**
	0.50	*0.50*	*0.45*	*0.45*	*0.45*
4.00	**0.87**	**0.88**	**0.87**	**0.83**	**0.78**
	0.85	*0.80*	*0.75*	*0.70*	*0.65*
5.00	**1.27**	**1.28**	**1.28**	**1.25**	**1.19**
	1.40	*1.30*	*1.20*	*1.10*	*1.00*

Panel C: PLC Expansion Option Value

Switch Mean	Switch Volatility *1.00*	*1.50*	*2.00*	*2.50*	*3.00*
3.00	**0.47**	**0.54**	**0.59**	**0.61**	**0.61**
	0.05	*0.00*	*0.00*	*0.00*	*0.00*
4.00	**1.34**	**1.38**	**1.38**	**1.36**	**1.32**
	0.15	*0.10*	*0.05*	*0.00*	*0.00*
5.00	**3.20**	**3.12**	**2.96**	**2.78**	**2.59**
	0.20	*0.15*	*0.15*	*0.10*	*0.05*

Panel D: PLC General Option Value

Switch Mean	Switch Volatility *1.00*	*1.50*	*2.00*	*2.50*	*3.00*
3.00	**1.23**	**1.24**	**1.20**	**1.15**	**1.10**
	0.15	*0.15*	*0.15*	*0.15*	*0.00*
4.00	**2.55**	**2.52**	**2.41**	**2.27**	**2.13**
	0.20	*0.20*	*0.15*	*0.15*	*0.15*
5.00	**4.83**	**4.69**	**4.42**	**4.10**	**3.76**
	0.20	*0.20*	*0.20*	*0.20*	*0.15*

Table 4

Sensitivity of Option Value to Production Modularity

Panel A lists the value of contraction options in bold and corresponding optimal initial capacities in italics as a function of overhead cost and salvage value. Panel B lists the value of expansion options in bold and corresponding optimal initial capacities in italics as a function of overhead cost and expansion cost. Expansion costs and salvage values are a percentage of per unit initial cost.

Panel A: PLC Contraction Option Value

Salvage Value	Overhead Cost *0.30*	*0.40*	*0.50*	*0.60*	*0.70*
0.75	**1.33**	**1.28**	**1.23**	**1.18**	**1.13**
	2.15	*1.90*	*1.65*	*1.50*	*1.35*
0.85	**1.55**	**1.48**	**1.41**	**1.34**	**1.27**
	2.25	*2.00*	*1.75*	*1.55*	*1.40*
0.95	**1.81**	**1.71**	**1.61**	**1.51**	**1.42**
	2.50	*2.15*	*1.90*	*1.65*	*1.50*

Panel B: PLC Expansion Option Value

Expansion Cost	Overhead Cost *0.30*	*0.40*	*0.50*	*0.60*	*0.70*
-1.00	**2.54**	**2.80**	**2.98**	**3.11**	**3.20**
	0.40	*0.35*	*0.30*	*0.25*	*0.20*
-1.25	**2.15**	**2.40**	**2.59**	**2.72**	**2.82**
	0.65	*0.50*	*0.40*	*0.35*	*0.30*
-1.50	**1.85**	**2.09**	**2.27**	**2.41**	**2.50**
	0.80	*0.65*	*0.55*	*0.45*	*0.40*

Figure 1

Semiconductor Supply History

Depicted below are aggregate shipments of DRAM semiconductor chips, in millions per year, categorized by chip memory. Data supplied by the Semiconductor Research Corporation.

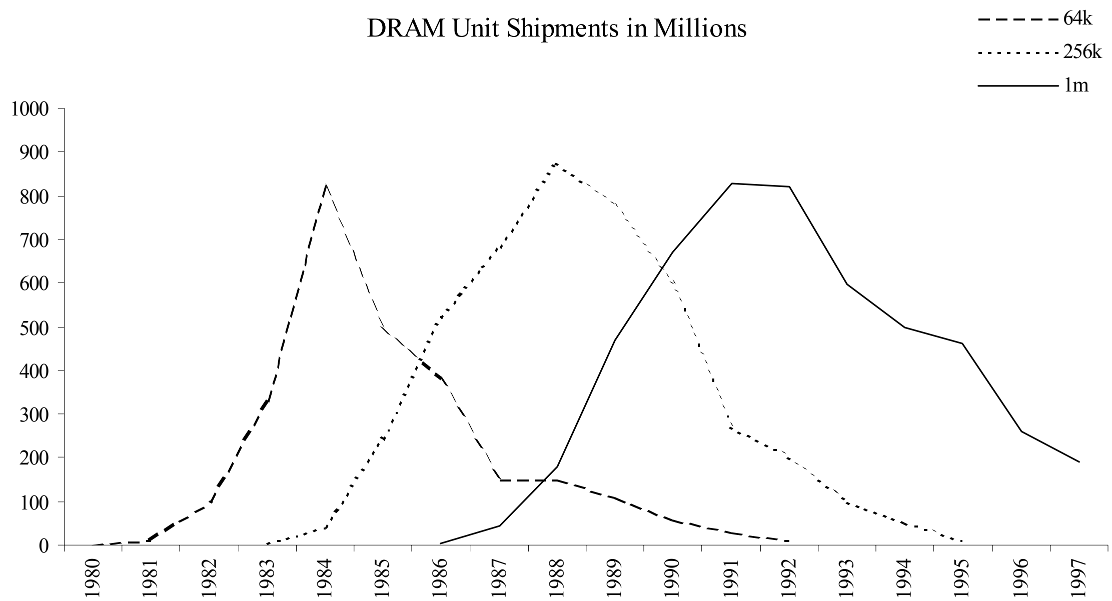
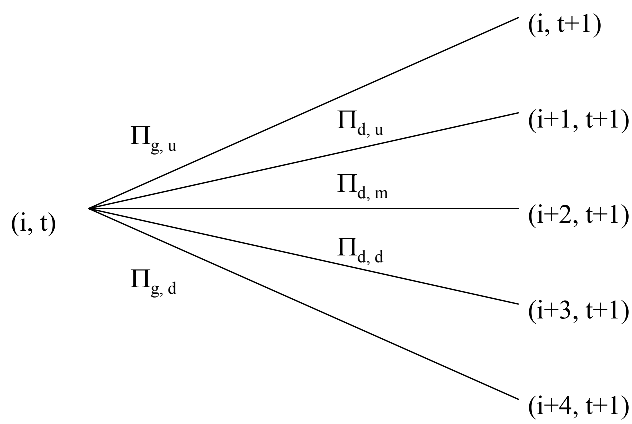


Figure 2

Pentanomial Lattice

The figure below shows a single time step of a pentanomial lattice. The outer two branches represent the growth regime. The inner three branches represent the decay regime. The step size of the outer distribution is given from the binomial lattice. The step size of the inner distribution is set equal to one-half the outer distribution's step size to allow for efficient recombining. The third branch is added to the inner distribution to maintain the match between the underlying distribution and the distribution implied by the lattice. The level of the underlying variable is indicated by *i* where *i* = 1 is the highest level in each time step.



Endnotes

[1] See, for example, Pindyck (1988) and Trigeogis (1996), p. 290.

[2] See, for example, Pindyck (1988) and Trigeogis (1996), p. 291.

[3] When firms achieve Cournot or von Stackelberg equilibrium, the firms' optimal production quantities are known functions of the demand parameter θ, as shown in Kreps (1990), p.326-329. In this setting the analysis can proceed as in the monopolist case, with the addition of a market share parameter that is determined by the number of firms and the nature of the equilibrium. The analysis cannot, however, easily accommodate competitive strategies that disrupt equilibrium.

[4] The discount rate used in equation (8) depends on assumptions about risk preferences. As discussed in section 2, it is often possible to discount at the riskless rate when the growth rate of demand is risk-adjusted.

[5] Discreteness is realistic when production capacity is a function of some integer number of machines or assembly lines. Discreteness is also a useful assumption because it keeps the problem tractable, as shown below.

[6] If the monopolist can determine the evolution of the product life cycle, perhaps by controlling the introduction of substitute products, then the duration of each state in the product life cycle is endogenous. The joint assumption of a monopoly and a stochastic product life cycle can be motivated by the notion that firms compete by developing new and improved products. The firm that introduces a superior product is rewarded by patent protection or a technological lead over its rivals, and hence can be viewed to some extent as a monopolist.

[7] Some product life cycles may have additional stages. The numerical technique described in section 2 can be extended to accommodate them.

[8] This assumption is necessary for valuing options with early exercise in a regime-switching model, as discussed in Bollen (1997).

[9] Those readers unfamiliar with the concept of risk-neutral discounting can find a non-technical explanation in Hull (1997), p. 288-292.

[10] It would be possible but more complicated to model capacity changes that take multiple periods to take effect.