How Much Information Is Required to Accurately Predict Merger Effects?

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Abstract

This paper poses the question "how much information is required to accurately predict merger effects?" To answer it, we develop a homotopy method for tracing a continuous path from the observed pre-merger equilibrium to the unobserved post-merger equilibrium. The path is defined by a sequence of "partial" mergers. We present a methodology for extrapolating along this homotopy path and, by considering information only at the observed pre-merger equilibrium, we identify parameters that determine merger effects. We show that the path is almost linear in price, which gives us a basis for arguing that the linear extrapolation is likely to give good estimates of post-merger prices, or equivalently, that information on demand first and second derivatives is enough, in most cases, to accurately predict merger effects. We present a closed-form merger predictor from which confidence intervals can be computed.

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1 Introduction

Since 1995, antitrust authorities in the United States (Werden [14]), and more recently Europe, have been formally modeling merger effects as the difference between two non-cooperative equilibria, one in which the merged products are priced independently, and one in which they are priced jointly. To asses the utility of the approach, it is important to ask how much information is required to implement it, *i.e.*, how much information is required to accurately predict merger effects? Since only the pre-merger equilibrium is observed, this question can be rephrased by asking "how much information at the observed pre-merger equilibrium is required to extrapolate to the unobserved post-merger equilibrium?"

Computing non-cooperative equilibrium requires information about the firm-level profit functions, the most important component of which is typically demand. Consequently, in what follows, we are going to focus on uncertainty about demand but most of our conclusions would apply to uncertainty about costs as well.

A technique useful for computing equilibrium uses information about the solution to a "nearby" problem to find a solution to a given problem. A homotopy method of solution assumes that problems are defined by a continuous parameter space and that solutions vary continuously with model parameters. A solution for one set of parameters, *e.g.,* the pre-merger equilibrium, is taken as a starting point. As the parameters are continuously varied to give new problems, the solutions of these intermediate problems define a path in the solution space.

Homotopy methods have a variety of uses. In some cases, it is easier to trace a path in solution space than to directly compute the solution of the new problem (*e.g.*, Judd [9]). In others, homotopy methods are used to find multiple equilibria (*e.g.*, Herings and van den Elzen [7]). In this paper, we use a homotopy method to extrapolate from pre- to post-merger equilibrium. By using information only at the pre-merger equilibrium, we are able to identify parameters that determine merger effects. We find that, contrary to current practice which uses only first derivatives of demand (elasticities), accurate prediction also requires information on second derivatives (curvature) as well. We develop a closed-form merger predictor based on estimated first and second derivatives of demand.

To make a homotopy method applicable to merger analysis we need a continuous transition from pre-merger to post-merger equilibrium. In this paper, we parameterize such a path along a sequence of "partial" mergers (Bresnahan and Salop [2] and O'Brien and Salop [10]), where we allow the merged entities to maximize a linear combination of own and rival-firm profits. For example, let r be the weight that a firm places on its rival's profits. A value of $r = 0$ corresponds to the pre-merger equilibrium, and a value of $r = 1$ corresponds to the post-merger equilibrium. As r increases from zero to one, this defines a continuous path from pre- to post-merger equilibrium.

In what follows, we first describe current practice, and then set up the theoretical machinery necessary to compute equilibrium with partial mergers. We then define a homotopy path and show how to extrapolate to the post-merger equilibrium based on the properties of the profit function at the pre-merger equilibrium. We show that the homotopy path is almost linear in price, which gives us a basis for arguing that linear extrapolation is likely to give good estimates of post-merger prices, or equivalently, that information on demand first and second derivatives is enough, in most cases, to accurately predict merger effects.

2 Computing Merger Effects Using Assumed Demand Forms

In this section, we examine methodologies used for predicting merger effects. They all require knowledge of the firm-level profit functions, which are typically constructed from estimated demand systems. We show that mistaken assumptions about demand functional form can lead to gross prediction errors.

2.1 Merger Effects in Bertrand Industries

The merging products, and those products whose prices change in response to changes in merging-product prices, are indexed by $i = 1, 2, \ldots, N$. As a function of the vector of prices $\mathbf{p} = \{p_i\}$, consumers demand quantities $\mathbf{q} = \{q_i\}$ where p_i and q_i are the price and quantity demanded of the i-th product. We assume that demand for the i-th product is supplied at a cost c_i which, for simplicity, is taken to be a function of q_i only. The profit on the i-th product is then

$$\pi_i = p_i q_i - c_i \tag{1}$$

If each product were controlled by a different firm whose only goal was to maximize profit on their product, then the first-order condition for a Nash equilibrium would be given by

$$0 = \frac{\partial \pi_i}{\partial p_i} = q_i + (p_i - mc_i)\frac{\partial q_i}{\partial p_i} \tag{2}$$

where

$$mc_i = \frac{dc_i}{dq_i} \tag{3}$$

is the marginal cost of product i. We need not assume that marginal costs are constant, though this is often done in practice. The partial derivative of q_i with respect to p_j is usually expressed in terms of elasticities

$$\epsilon_{ij} = \frac{p_j}{q_i}\frac{\partial q_i}{\partial p_j} \tag{4}$$

so the first-order condition implies

$$\frac{p_i - mc_i}{p_i} = -\frac{1}{\epsilon_{ii}}, \tag{5}$$

the familiar relation between price-cost margin and own-price elasticity ϵ_{ii}.

When different products are controlled by the same firm, the firm acts to internalize the price effects on its jointly-owned products. For example, if a firm controls both products one and two, then the first-order conditions for that firm are

$$0 = \frac{\partial(\pi_1 + \pi_2)}{\partial p_1} = q_1 + (p_1 - mc_1)\frac{\partial q_1}{\partial p_1} + (p_2 - mc_2)\frac{\partial q_2}{\partial p_1} \tag{6}$$

and

$$0 = \frac{\partial(\pi_1 + \pi_2)}{\partial p_2} = q_2 + (p_1 - mc_1)\frac{\partial q_1}{\partial p_2} + (p_2 - mc_2)\frac{\partial q_2}{\partial p_2} \tag{7}$$

which leads to a different equilibrium pricing when the cross-price elasticities are nonzero. It is the difference between these two equilibria that is the "unilateral" effect of a merger (*Horizontal Merger Guidelines* [13]), so called because it does not require "coordinated" pricing with the non-merging firms.

2.2 Computing Merger Effects

In computing profit functions, economists typically estimate demand using an assumed functional form and recover the marginal costs from the moment restriction implied by the first-order conditions at the pre-merger equilibrium, *e.g.*, Equation (5).

The simplest approaches to computing merger effects are based on a constant-elasticity approximation to an unknown demand curve (*e.g.*, Shapiro [12]). Using an estimate of the elasticity matrix at the pre-merger equilibrium, it is possible to analytically compute post-merger prices, *e.g.,* from equations (6) and (7). The constant-elasticity specification has the virtue of not requiring information about non-merging products as they do not change price in response to a merger. However, if demand becomes more elastic as price increases, this methodology can over-estimate price changes by several orders of magnitude (Werden and Froeb, [16]).

INSERT Figure 1 and Figure 2 ABOUT HERE

This problem can be addressed, though not solved, by using demand systems that become more elastic as prices rise, systems such as the Almost Ideal Demand System (AIDS) or nested logit. However, the functional form still determines, to a large extent, the predicted post-merger prices (Crooke *et. al.*, [3]). This is illustrated in Figure 1 where we plot five aggregate demand curves between the competitive and monopoly equilibrium. This can be thought of as a very primitive merger model, where a merger moves an industry from competition to monopoly. All five demand curves are calibrated to the same pre-merger equilibrium, where $price = \$4$, $quantity = 10$, and $elasticity = -2$. The predicted post-merger price rise is largest for the constant elasticity specification (CE) followed by the AIDS, logit, probit and then linear specification. Depending on the functional form, predicted price effects can differ by a factor of four.

The reason for the differences among the functional forms is related to how elasticity changes as price changes. This dependence is illustrated in Figure 2 where the same five demand curves are plotted in price/elasticity space rather than in price/quantity space. As is evident, the price changes are smaller for demand curves whose elasticity becomes larger as price increases. For the constant-elasticity specification, elasticity does not change, but for the logit, probit, and linear specifications, elasticity more than doubles.

3 Partial Mergers

In this section, we describe a path from pre- to post-merger equilibrium by considering a sequence of "partial" mergers (O'Brien and Salop [10]). To compute equilibrium in a partially-merged industry, we imagine that a different agent controls the pricing of each product but that each agent may share in the profits of other products. This simple algorithm allows us to characterize different kinds of equilibria, including the pre-merger equilibrium, where prices are set independently, and the post-merger equilibrium where the merged entity maximizes the sum of profits on its jointly-owned products.

To make this notion precise, let $\mathbf{W} = \{w_{ij}\}$ specify the ownership structure where we take w_{ij} to be the share of profit on product j received by the agent who sets price on product i, so that this agent i maximizes

$$\Omega_i = \sum_j w_{ij} \pi_j. \tag{8}$$

We call the N by N matrix $\mathbf{W}$ the ownership-structure matrix or simply the ownership matrix. We could require, for each j, $\sum_i w_{ij} = 1$, to represent that all profits are distributed, but this condition isn't necessary since multiplying all coefficients for a given i by the same positive constant, multiplies Ω_i by that constant, and thus does not affect the optimal pricing of product i. If Λ is a diagonal matrix, then ownership matrices $\mathbf{W}$ and $\Lambda\mathbf{W}$ are to be considered equivalent since they lead to the same Nash equilibrium. The first-order conditions are then, for each i,

$$0 = \frac{\partial \Omega_i}{\partial p_i} = \sum_j w_{ij} \frac{\partial \pi_j}{\partial p_i} = w_{ii} q_i + \sum_j w_{ij} (p_j - mc_j) \frac{\partial q_j}{\partial p_i}. \tag{9}$$

The premerger scenario where each product is owned by a different firm is represented by $\mathbf{W}$ equal to the identity matrix. A partial merger between firms controlling products one and two in a three-product industry can be represented by a matrix which is the identity except for the off-diagonal entries w_{12} and w_{21},

$$\mathbf{W}(\mathbf{r}) = \begin{pmatrix} 1 & r & 0 \\ r & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{10}$$

where r is a parameter varying between 0 and 1 representing the extremes of the pre- and post-merger scenarios.

There is no reason to assume that the ownership matrix $\mathbf{W}$ is symmetric. It may well be that firm one buys a less-than-50% interest in firm two and product one is priced with this ownership arrangement in mind while firm two continues to price product two as an independent product.

INSERT Figure 3 and Figure 4 ABOUT HERE

In Figures 3 and 4, we illustrate the path in the solution space of equilibria determined from the ownership matrix in Equation (10) from pre- to post-merger equilibrium for a logit (*e.g.,* Werden and Froeb [15]) and for an Almost Ideal Demand System (AIDS) without income effects (*e.g.,* Hausman, Leonard, and Zona [6]), both calibrated to the same pre-merger prices, shares, and elasticities. Firms have pre-merger shares of 45%, 20%, and 35%, and pre-merger prices of \$45, \$20, and \$35, respectively. Prices are set equal to shares so we can graph price and quantity changes on the same graph. We consider a merger between firms one and two.

Using the parameterization in Werden and Froeb [15], we generate an elasticity matrix from a logit demand with an aggregate demand elasticity of -2.0 and a price coefficient of $-.15$,

$$elasticity = \begin{pmatrix} -4.82 & 0.38 & 1.17 \\ 1.93 & -2.62 & 1.17 \\ 1.93 & 0.38 & -4.08 \end{pmatrix} \tag{11}$$

where quantities are in rows and prices are in columns. Note that the cross-price elasticities are the same in a given column and that big firms are good substitutes for small ones, but not *vice-versa*. These two features are a consequence of the Independent of Irrelevant Alternatives (IIA) characteristic of logit demand.

In Figure 3, we illustrate the path from pre- to post-merger equilibrium defined by $\mathbf{W}(r)$ in Equation (10). As r goes from zero to one along the horizontal axis, we see that both merging prices increase and both merging quantities decrease, with the smallest merging firm increasing prices by the most. There is a price increase and a quantity gain by the non-merging firm. These changes imply a reallocation of output from the merging to the non-merging firms, and from the smaller merging firm to its larger merging partner.

The same kind of output reallocations are evident in Figure 4 which shows a pre- to post-merger equilibrium path for an AIDS demand system calibrated to the same elasticity matrix (see Crooke *et. al.* [3] for calibration details). What is significant about both of these pictures is that the price path is almost linear in r. This suggests that extrapolation based on local demand information at the pre-merger equilibrium should fit quite well.

3.1 Newton's Method for Computing Post-merger Equilibrium

Before considering uncertainty about demand, and consequently uncertainty about the firm-level profit functions, we are first going to assume that the profit functions are known and review how one would ordinarily find a Nash equilibrium satisfying the first-order conditions (9) using Newton's method. Let $z_i = \partial\Omega_i/\partial p_i$ and let $\mathbf{z} = \{z_i\}$ be the vector of functions of $\mathbf{p}$. To solve for simultaneous zeros of the z_i we take an initial estimate $\mathbf{p}^{(0)}$ and refine that estimate by taking $\mathbf{p}^{(1)}$, $\mathbf{p}^{(2)}$, $\mathbf{p}^{(3)} \ldots$, defined by

$$\mathbf{p}^{(k+1)} = \mathbf{p}^{(k)} - \mathbf{A}^{-1}\mathbf{z}^{(k)} \tag{12}$$

where $\mathbf{z}^{(k)} = \mathbf{z}(\mathbf{p}^{(k)})$ is the value of the objective functions at the latest approximation, and $\mathbf{A} = \{a_{ij}\}$ where

$$a_{ij} = \frac{\partial z_i}{\partial p_j} = w_{ii}\frac{\partial q_i}{\partial p_j} + w_{ij}\frac{\partial q_j}{\partial p_i} + \sum_k w_{ik}(p_k - mc_k)\frac{\partial^2 q_k}{\partial p_j \partial p_i} \tag{13}$$

is also evaluated at $\mathbf{p}^{(k)}$, assuming for simplicity that mc_k is constant. The matrix $\mathbf{A}$ is simply a derivative and we may write

$$\mathbf{A} = \frac{\partial \mathbf{z}}{\partial \mathbf{p}} \tag{14}$$

Then Newton's method is seen as solving the first-order approximation to the $\mathbf{z}$ at $\mathbf{p}^{(k)}$

$$\mathbf{z} \approx \mathbf{z}^{(k)} + \frac{\partial \mathbf{z}}{\partial \mathbf{p}} \left(\mathbf{p} - \mathbf{p}^{(k)} \right) \tag{15}$$

for the value of $\mathbf{p}$ that makes the approximation to $\mathbf{z}$ zero.

Usually a problem in Newton's method is finding a reasonable starting point $\mathbf{p}^{(0)}$. In merger analysis, a natural starting point is the pre-merger equilibrium. Taking $\mathbf{z}^{(0)}$ to be the first-order conditions evaluated with the post-merger ownership matrix at the pre-merger equilibrium prices, the first step from Newton's method is $\mathbf{p}^{(1)} = \mathbf{p}^{(0)} + \mathbf{A}^{-1}\mathbf{z}^{(0)}$. The approximation can be refined by iterating Newton's method, but this requires information about the profit functions at prices away from the initial price vector.

Applying Newton's method requires us to be able to evaluate not just $\mathbf{z}$ but also the derivatives of $\mathbf{z}$. Since $\mathbf{z}$ depends on the first derivatives of the demand functions, the derivatives of $\mathbf{z}$ will be expressed in terms of second derivatives of $\mathbf{q}$. The second derivatives measure how narrow the peak is in firm profit functions or, put another way, how much it costs firms to deviate from equilibrium pricing.

4 A Homotopy Method for Computing Post-merger Equilibrium

Note that the homotopy path along the sequence of ownership structures defined by Equation (10) is not unique – there are an infinite number of paths from pre- to post-merger equilibrium. For example, we could parameterize a path in $s = 2r/(1+r)$ defined by

$$\mathbf{W}(s) = \begin{pmatrix} 1 - s/2 & s/2 & 0 \\ s/2 & 1 - s/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{16}$$

for s varying between 0 and 1. The path in s would correspond to firms trading their own shares for rival shares. Since the path in r is nearly linear, the path in s is not because s is a nonlinear function of r.

In this section, we parameterize two different paths from pre- to post-merger, and then extrapolate along the paths using information only at the pre-merger equilibrium. Extrapolating along a path is useful for understanding how merger effects depend on the local characteristics of the demand system. We use the extrapolation to derive a predictor of post-merger prices that depends only on the characteristics of demand at the observed pre-merger equilibrium.

4.1 A Linear Approximation

To apply homotopy methods to finding Nash equilibrium solutions to the first-order conditions specified in equation (9), we assume that the equilibrium prices are a differentiable function $\hat{\mathbf{p}}(\mathbf{W})$ of the ownership matrix $\mathbf{W}$ in some region. We find a linear system of equations for the partial derivatives of the $\hat{p}_k$ with respect to the matrix entries $w_{\ell m}$ by differentiating equation (9) with respect to $w_{\ell m}$ keeping in mind that profits are functions of prices which are functions of the ownership matrix. For each i,

$$0 = \frac{\partial}{\partial w_{\ell m}} \left(z_i|_{\mathbf{p}=\hat{\mathbf{p}}(\mathbf{W})} \right) = \frac{\partial z_i}{\partial w_{\ell m}} + \sum_k \frac{\partial z_i}{\partial p_k} \frac{\partial \hat{p}_k}{\partial w_{\ell m}} \tag{17}$$

It is useful to think of ℓm as a single index over N^2 elements, taking the ownership matrix $\mathbf{W}$ to be written as an N^2-dimensional column vector $\mathbf{w}$, and defining $\partial \mathbf{z} / \partial \mathbf{w}$ as an N by N^2 matrix (rather than a rank 3 tensor) by setting the entry in the i-th row and (lm)-th column to be

$$\frac{\partial z_i}{\partial w_{\ell m}} = \delta_{i\ell}\delta_{im} q_i + \delta_{i\ell}(p_m - mc_m) \frac{\partial q_m}{\partial p_i} \tag{18}$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise, the Kronecker delta. Write $\partial \hat{\mathbf{p}} / \partial \mathbf{w}$ for the N by N^2 matrix of ownership-structure sensitivity coefficients with k-th row and (ℓm)-th column given by $\partial \hat{p}_k / \partial w_{\ell m}$. Then the linear system (17) can be rewritten as

$$0 = \frac{\partial \mathbf{z}}{\partial \mathbf{w}} + \frac{\partial \mathbf{z}}{\partial \mathbf{p}} \frac{\partial \hat{\mathbf{p}}}{\partial \mathbf{w}} = \frac{\partial \mathbf{z}}{\partial \mathbf{w}} + \mathbf{A} \frac{\partial \hat{\mathbf{p}}}{\partial \mathbf{w}} \tag{19}$$

with $\mathbf{A}$ as before, and we get

$$\frac{\partial \hat{\mathbf{p}}}{\partial \mathbf{w}} = -\mathbf{A}^{-1} \frac{\partial \mathbf{z}}{\partial \mathbf{w}} \tag{20}$$

Note that $\partial \mathbf{z} / \partial \mathbf{w}$ is independent of $\mathbf{w}$ and most of its entries are 0, in particular when $i \neq \ell$.

Suppose now that we wish to evaluate equilibrium prices along a particular path in the ownership structure coefficient space. Suppose $\mathbf{W} = \mathbf{W}(r)$ is the ownership structure matrix defined for a parameter r varying between 0 and 1. Let $\hat{\mathbf{p}}(r) = \hat{\mathbf{p}}(\mathbf{W}(r))$. Write $d\mathbf{w}/dr$ and $d\hat{\mathbf{p}}/dr$ for the derivatives of the vector ownership structure coefficients $\mathbf{w}$ and the equilibrium prices $\hat{\mathbf{p}}$ with respect to r. The equilibrium price functions will satisfy

$$\frac{d\hat{\mathbf{p}}}{dr} = \frac{\partial \hat{\mathbf{p}}}{\partial \mathbf{w}}\frac{d\mathbf{w}}{dr} = -\mathbf{A}^{-1}\frac{\partial \mathbf{z}}{\partial \mathbf{w}}\frac{d\mathbf{w}}{dr} \tag{21}$$

Our initial state is defined by an ownership structure matrix $\mathbf{W}^{(0)} = \mathbf{W}(0)$ leading to given equilibrium prices $\mathbf{p}^{(0)} = \hat{\mathbf{p}}(0)$ so equation (21) defines an initial value problem to solve for $\hat{\mathbf{p}}$.

To the extent that the right-hand side of the differential equation (21) is nearly constant over the range of prices between $\hat{\mathbf{p}}(0)$ and $\hat{\mathbf{p}}(1)$ we can take the linear approximation

$$\hat{\mathbf{p}}(1) \approx \hat{\mathbf{p}}(0) - \mathbf{A}^{-1}\frac{\partial \mathbf{z}}{\partial \mathbf{w}}\frac{d\mathbf{w}}{dr} \tag{22}$$

Note that $(\partial \mathbf{z}/\partial \mathbf{w})\mathbf{w} = \mathbf{z}$ reflecting the fact that the $w_{\ell m}$ enter linearly in $\mathbf{z}$. Thus if $d\mathbf{w}/dr$ is constant, then $d\mathbf{w}/dr = \mathbf{w}(1) - \mathbf{w}(0)$, and we have

$$\frac{\partial \mathbf{z}}{\partial \mathbf{w}}\frac{d\mathbf{w}}{dr} = \frac{\partial \mathbf{z}}{\partial \mathbf{w}}(\mathbf{w}(1) - \mathbf{w}(0)) = \frac{\partial \mathbf{z}}{\partial \mathbf{w}}\mathbf{w}(1) = \mathbf{z}^{(0)} \tag{23}$$

where $\mathbf{z}^{(0)}$ is the first-order condition values for the post-merger structure coefficients at the initial prices as in Newton's method (12). Then $\hat{\mathbf{p}}(1) = \mathbf{p}^{(1)}$ is the same result as the first iteration of Newton's method, when $\mathbf{A}$ is evaluated with the post-merger structure coefficients. The condition that the right-hand side of the differential equation (21) is approximately constant is not guaranteed to hold simply by taking the firm structure coefficients varying linearly in r. That we can arrange to make the price functions approximately linear in r, so that the simple linear extrapolation is coincident with a single step of Newton's method, will be a byproduct of a more detailed analysis below.

Note that Equation (22) yields a closed-form predictor of merger price effects,

$$\mathbf{\Delta}\hat{\mathbf{p}} = \mathbf{A}^{-1}\mathbf{z}^{(0)}. \tag{24}$$

Equation (24) can be used in lieu of re-sampling techniques to directly compute confidence intervals around merger effects using the δ-method. If a demand system is parameterized such that $\mathbf{A}^{-1}\mathbf{z}^{(0)} = \mathbf{g}(\theta)$, and $\hat{\theta}$ is an estimator of θ with a limiting normal distribution $\sqrt{n}(\hat{\theta} - \theta) \overset{D}{\to} N(\mathbf{0}, \Sigma)$ then $\sqrt{n}(\mathbf{g}(\hat{\theta}) - \mathbf{g}(\theta)) \overset{D}{\to} N(\mathbf{0}, \nabla\mathbf{g}.\Sigma.\nabla\mathbf{g}')$, provided that $\mathbf{g}$ is continuously differentiable at θ. Such analytic expressions are useful for identifying factors that make confidence intervals large, and may suggest estimation strategies to reduce sampling variance.

4.2 A Quadratic Approximation

In the previous section, we considered a linear approximation to the solution of the differential equation (21) giving the equilibrium prices between pre- and post-merger ownership structure matrices $\mathbf{W}^{(0)}$ and $\mathbf{W}^{(1)}$. Substituting a quadratic approximation to the solution of the differential equation gives a better estimate for the merger effects. At the same time, we may alternatively consider a linear path between $\mathbf{W}^{(0)}$ and $\Lambda\mathbf{W}^{(1)}$ for a diagonal matrix Λ. This gets us to the same post-merger equilibrium but parameterizes the partial mergers differently. By judicious choice of the matrix Λ we can eliminate the second-order terms in the equilibrium price solution so that the linear approximations of the previous section apply. An approximation using only first and second derivatives of the demand system makes the second-order terms small and gives us a basis for arguing that the linear extrapolation is likely to give good estimates of post-merger prices, or equivalently, that information on demand first and second derivative is enough, in most cases, to accurately predict merger effects.

The equilibrium prices for ownership structure $\mathbf{W}^{(1)}$ will be the same as those for $\Lambda\mathbf{W}^{(1)}$ for any diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$. Define $\mathbf{W}$ as a linear function of r varying between 0 and 1 with $\mathbf{W}(0) = \mathbf{W}^{(0)}$ and $\mathbf{W}(1) = \Lambda\mathbf{W}^{(1)}$, namely take

$$w_{ij}(r) = (1-r)w_{ij}^{(0)} + r\lambda_i w_{ij}^{(1)} \tag{25}$$

Take $\hat{\mathbf{p}}(r) = \hat{\mathbf{p}}(\mathbf{W}(r))$ to be the solution of the first-order conditions (9) with ownership structure matrix $\mathbf{W}(r)$. To get an estimate of prices at the post-merger equilibrium, we consider a series expansion of $\hat{\mathbf{p}}(r)$:

$$\hat{\mathbf{p}}(1) \approx \hat{\mathbf{p}}(0) + \frac{d\hat{\mathbf{p}}}{dr} + \frac{1}{2}\frac{d^2\hat{\mathbf{p}}}{dr^2} \tag{26}$$

We will use the additional degrees of freedom in choosing Λ to make the equilibrium solution function $\hat{\mathbf{p}}$ approximately linear in r at $r = 0$. Our strategy is to determine the first and second derivatives of $\hat{\mathbf{p}}$ with respect to r at $r = 0$. We evaluate the first and second derivatives of the first-order conditions $\mathbf{z}|_{\mathbf{p}=\hat{\mathbf{p}}}$ at $r = 0$ in terms of the derivatives of the demand system, $\mathbf{W}$, and $\hat{\mathbf{p}}$. Since $\mathbf{z}|_{\mathbf{p}=\hat{\mathbf{p}}} = \mathbf{0}$, we can solve these constraints for the unknown first and second derivatives of $\hat{\mathbf{p}}$ thus getting a second-order approximation to the solution of the differential equation (21) at $r = 0$.

To solve for first derivative, $d\hat{\mathbf{p}}/dr$ at $r = 0$, we get

$$0 = \frac{d}{dr}\left(\mathbf{z}|_{\mathbf{p}=\hat{\mathbf{p}}}\right) = \frac{\partial \mathbf{z}}{\partial \mathbf{w}}\frac{d\mathbf{w}}{dr} + \frac{\partial \mathbf{z}}{\partial \mathbf{p}}\frac{d\hat{\mathbf{p}}}{dr} = \frac{\partial \mathbf{z}}{\partial \mathbf{w}}\frac{d\mathbf{w}}{dr} + \mathbf{A}\frac{d\hat{\mathbf{p}}}{dr}. \tag{27}$$

The term $d\hat{\mathbf{p}}/dr$ at $r = 0$ is a function of $\mathbf{A}$ and $\partial\mathbf{z}/\partial\mathbf{w}$ evaluated at $r = 0$.

To solve for the second derivative, note that $d^2 w_{ij}/dr^2 = 0$, and that $\partial z_i/\partial w_{ij}$ is independent of $\mathbf{W}$. This leads to the equation,

$$\begin{aligned} 0 &= \frac{d^2}{dr^2}\left(z_i|_{\mathbf{p}=\hat{\mathbf{p}}}\right) \qquad (28) \\ &= 2\sum_{j,k}\frac{\partial^2 z_i}{\partial w_{ij}\partial p_k}\frac{dw_{ij}}{dr}\frac{d\hat{p}_k}{dr} + \sum_{k,\ell}\frac{\partial^2 z_i}{\partial p_\ell \partial p_k}\frac{d\hat{p}_k}{dr}\frac{d\hat{p}_\ell}{dr} + \sum_k \frac{\partial z_i}{\partial p_k}\frac{d^2\hat{p}_k}{dr^2} \end{aligned}$$

With $d\hat{\mathbf{p}}/dr$ at $r = 0$ from Equation (27), we solve Equation (28) for $d^2\hat{\mathbf{p}}/dr^2$ at $r = 0$.

Suppose the second term in Equation (28) is zero. We will solve for Λ so that the first term is zero. Then $d^2\hat{p}_k/dr^2$ will be zero at $r = 0$ (as $\partial z_i/\partial p_k$ is not usually zero). That is, we will have parameterized the solution functions so that the second order terms vanish at $r = 0$ and a linear extrapolation will give a good approximation to the post-merger equilibrium.

Take λ_i so that

$$
\begin{aligned}
0 &= \sum_{j,k} \frac{\partial^2 z_i}{\partial w_{ij} \partial p_k} \frac{d w_{ij}}{dr} \frac{d\hat{p}_k}{dr} && (29) \\
&= \sum_{j,k} \frac{\partial^2 z_i}{\partial w_{ij} \partial p_k} \frac{d\hat{p}_k}{dr} \left(\lambda_i w_{ij}^{(1)} - w_{ij}^{(0)} \right) && (30)
\end{aligned}
$$

implying

$$
\lambda_i = \frac{\sum_j w_{ij}^{(0)} u_{ij}}{\sum_j w_{ij}^{(1)} u_{ij}} \tag{31}
$$

where

$$
u_{ij} = \sum_k \frac{\partial^2 z_i}{\partial w_{ij} \partial p_k} \frac{d\hat{p}_k}{dr} \tag{32}
$$

This depends on $d\hat{\mathbf{p}}/dr$ but we can substitute back into the first derivative constraint (27) evaluated at $r = 0$ where $\sum_j w_{ij}^{(0)} \partial z_i / \partial w_{ij} = z_i = 0$ so

$$
0 = \sum_j \frac{\partial z_i}{\partial w_{ij}} \frac{d w_{ij}}{dr} + \sum_k \frac{\partial z_i}{\partial p_k} \frac{d\hat{p}_k}{dr} = \sum_j \lambda_i w_{ij}^{(1)} \frac{\partial z_i}{\partial w_{ij}} + w_{ij}^{(0)} u_{ij} \tag{33}
$$

to get

$$
0 = \sum_j w_{ij}^{(1)} \frac{\partial z_i}{\partial w_{ij}} + w_{ij}^{(1)} u_{ij} = \sum_j w_{ij}^{(1)} \frac{\partial z_i}{\partial w_{ij}} + \sum_{j,k} w_{ij}^{(1)} \frac{\partial^2 z_i}{\partial w_{ij} \partial p_k} \frac{d\hat{p}_k}{dr} \tag{34}
$$

This relates the value and derivative of the first-order conditions evaluated with post-merger structure coefficients at the initial equilibrium prices. That is

$$
\frac{d\hat{\mathbf{p}}}{dr} = -\mathbf{A}^{-1} \mathbf{z}^{(0)} \tag{35}
$$

where

$$
\mathbf{z}^{(0)} = \sum_j w_{ij}^{(1)} \frac{\partial z_i}{\partial w_{ij}}. \tag{36}
$$

Equation (36) evaluated at $r = 0$ is the initial function value in Newton's method (12) and the matrix $\mathbf{A}$ has entries

$$a_{ik} = \frac{\partial z_i^{(0)}}{\partial p_k} = \sum_j w_{ij}^{(1)} \frac{\partial z_i}{\partial w_{ij} \partial p_k} \tag{37}$$

evaluated at $r = 0$ also as in Newton's method. From the $d\hat{\mathbf{p}}/dr$ we determine the u_{ij} and so the λ_i by (31). Fix these λ_i and take $d\hat{\mathbf{p}}/dr$ at $r = 0$ to have been determined by this method.

In general, the second partials of the first-order conditions with respect to prices, *i.e.* the second term in Equation (28), are not quite zero so we cannot have the $d^2\hat{\mathbf{p}}/dr^2$ in (28) exactly zero. With the λ_i taken to have the first term of (28) drop out and $d^2\hat{\mathbf{p}}/dr^2$ at $r = 0$ already determined we must have $d^2\hat{\mathbf{p}}/dr^2$ so

$$0 = \sum_{k,\ell} \frac{\partial^2 z_i}{\partial p_\ell \partial p_k} \frac{d\hat{p}_k}{dr} \frac{d\hat{p}_\ell}{dr} + \sum_k \frac{\partial z_i}{\partial p_k} \frac{d^2 \hat{p}_k}{dr^2} \tag{38}$$

Take $\mathbf{b} = \{b_i\}$ with entries defined by

$$b_i = \sum_{k,\ell} \frac{\partial^2 z_i}{\partial p_\ell \partial p_k} \frac{d\hat{p}_k}{dr} \frac{d\hat{p}_\ell}{dr} \tag{39}$$

so (38) becomes

$$0 = \mathbf{b} + \mathbf{A} \frac{d^2 \hat{\mathbf{p}}}{dr^2} \tag{40}$$

and so we get that $d^2\hat{\mathbf{p}}/dr^2 = -\mathbf{A}^{-1}\mathbf{b}$. The second-order approximation to $\hat{\mathbf{p}}$ gives a new equilibrium of

$$\hat{\mathbf{p}}(1) \approx \mathbf{p}^{(0)} + \frac{d\hat{\mathbf{p}}}{dr} + \frac{1}{2} \frac{d^2 \hat{\mathbf{p}}}{dr^2} \tag{41}$$

refining the one step Newton's method approximation $\hat{\mathbf{p}}^{(1)} = \mathbf{p}^{(0)} + d\hat{\mathbf{p}}/dr$.

The refinement term depends in general on third partials of the demand. These third derivatives may often be much smaller than the second derivative terms in the refinement term. If the higher-order terms are negligable or if we are only interested in the equilibrium computation for a demand system approximated to second-order terms then we may compute the refinement term neglecting the higher-order terms in $\mathbf{b}$.

4.3 Example

Using the analysis of the previous section, we can trace a path from pre- to post-merger equilibrium that has little or no curvature at the pre-merger equilibrium. When we extrapolate to the post-merger equilibrium, the line will give good estimates of the merger effect. To see how important this is, compare the extrapolations implicit in Figure 1 to those illustrated in Figures 3 and 4. In Figure 1, even though all five demand curves have the same first derivative at the pre-merger equilibrium, their curvatures are different. The different curvatures lead to different predicted merger effects. In contrast, the lines in Figures 3 and 4 have curvatures near zero, so linear extrapolation works much better. In this section, we give a heuristic explanation for the almost linear price paths in Figures 3 and 4.

For the pre-merger ownership structure, each product is separately owned, and for the post-merger ownership structure $\mathbf{W}^{(1)}$ is a diagonal matrix except for entries $w_{12}^{(1)} = w_{11}^{(1)}$ and $w_{21}^{(1)} = w_{22}^{(1)}$. Then

$$\lambda_1 = \frac{w_{11}^{(0)} u_{11}}{w_{11}^{(1)} u_{11} + w_{12}^{(1)} u_{12}} = \frac{w_{11}^{(0)} u_{11}}{w_{11}^{(1)} (u_{11} + u_{12})} \tag{42}$$

But u_{11} is roughly the change in $\partial\pi_1/\partial p_1$ while u_{12} is roughly the change in $\partial\pi_2/\partial p_1$ from pre- to post-merger. Pre-merger $\partial\pi_1/\partial p_1 = 0$ and $\partial\pi_2/\partial p_1$ is of some magnitude. To restore equilibrium, post-merger $\partial\pi_1/\partial p_1 + \partial\pi_2/\partial p_1 = 0$. We don't expect that the sensitivity of profit two to the price one will vary much as prices adjust, so $\partial\pi_2/\partial p_1$ will be relatively small pre- and post-merger. Instead, most of the change should come in the sensitivity of the profit on product one to its own price. Intuitively u_{11} will be much larger than u_{12} and so $\lambda_1 \approx w_{11}^{(0)}/w_{11}^{(1)}$, and similarly for the other λ_i. That is, we should expect that the diagonal of $\Lambda\mathbf{W}^{(1)}$ will be approximately the same as $\mathbf{W}^{(0)}$. Thus if $\mathbf{W}^{(0)}$ is the identity matrix then $\Lambda\mathbf{W}^{(1)}$ will have a corner 2×2 matrix of ones, and the intermediate matrix $\mathbf{W}(r)$ is given by Equation (10). In other words, the quadratic extrapolation of Section 4.2, with no curvature at the pre-merger equilibrium, is very close to the price paths in Figure 3 and Figure 4. This explains why equilibrium prices look almost linear in r, where r is taken to be the weight that the merging firms place on rival profits.

5 Conclusion

We can now answer the question posed in the title of the paper. To assess the effects of mergers, we need information on demand second derivatives (curvature), in addition to information about demand first derivatives (elasticities). The analysis of Section 4.2 shows how higher-order terms in the expansion of the demand system enter into the approximation of a solution and lead us to the conclusion that first and second derivatives of the demand system will suffice to accurately determine merger effects in most cases.

We present a merger predictor based on the linear extrapolation in Section 4.1 that is equivalent to the first step in Newton's method. The closed-form predictor allows computation of standard errors as a function of demand uncertainty. Without standard errors, it is difficult to evaluate the merits of competing point estimates provided by opposing parties in a merger case.

The analysis focuses attention on the problem of estimating demand curvature. It is easy to specify demand forms with flexible curvature, for example, using a Box-Cox transform (Crooke *et. al.* [3]) or a polynomial logit (Saha and Simon [11]), but attention needs to be given to estimation strategies and testing how well they work with available data. More work needs to be done on random coefficients logit models (Berry, Levinsohn, and Pakes [1]) to determine whether they can identify demand curvature.

In the absence of good estimates of demand curvature, sensitivity analysis is indicated. To do this, one should compute merger effects with different assumed demand forms, calibrated to the same data, *e.g.* as in Crooke *et. al.* [3].

Alternatively, "robust" methods for analyzing mergers may be used (Werden [14] and Froeb and Werden [5]). With this approach, marginal cost reductions sufficient to offset the price effects of the merger are computed from post-merger first-order conditions. Since prices do not change, neither does demand curvature. However, this approach is useful only for analyzing mergers under a consumer welfare standard. To assess merger effects under a total welfare standard, or to compute complex post-merger scenarios (*e.g.*, Jayaratne and Shapiro [8]), information about demand curvature would still be necessary.

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6 FIGURES

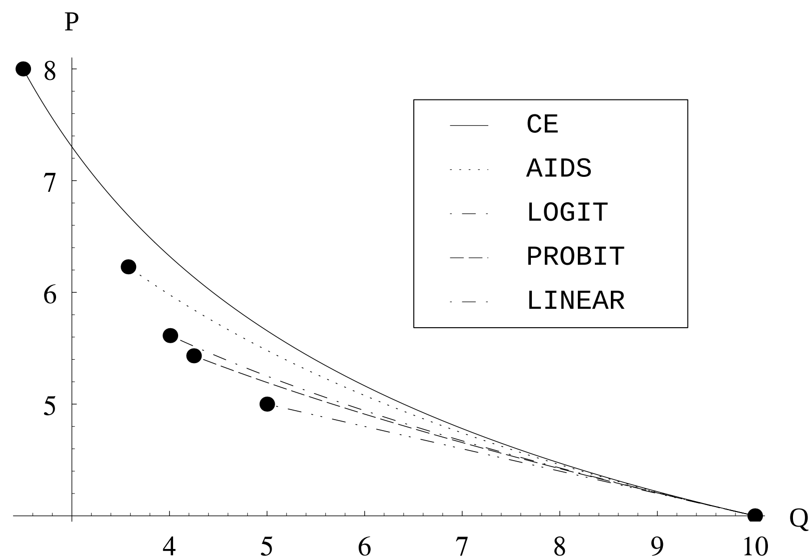

Figure 1: Demand Between Competition and Monopoly

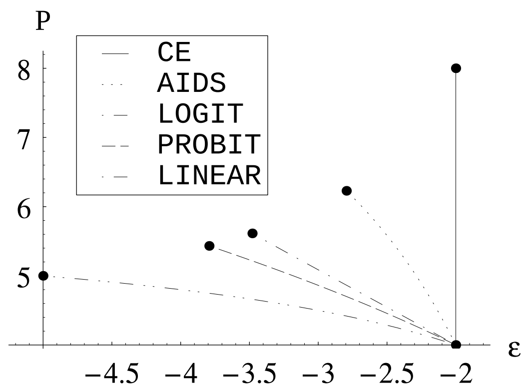

Figure 2: Elasticity Between Competition and Monopoly

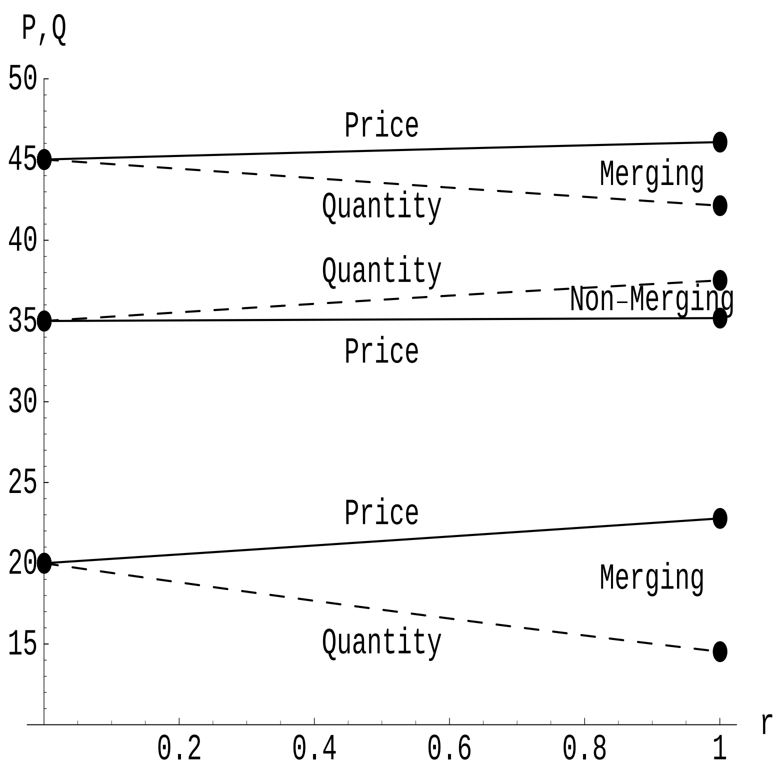

Figure 3: Merger Homotopy Path with Logit Demand

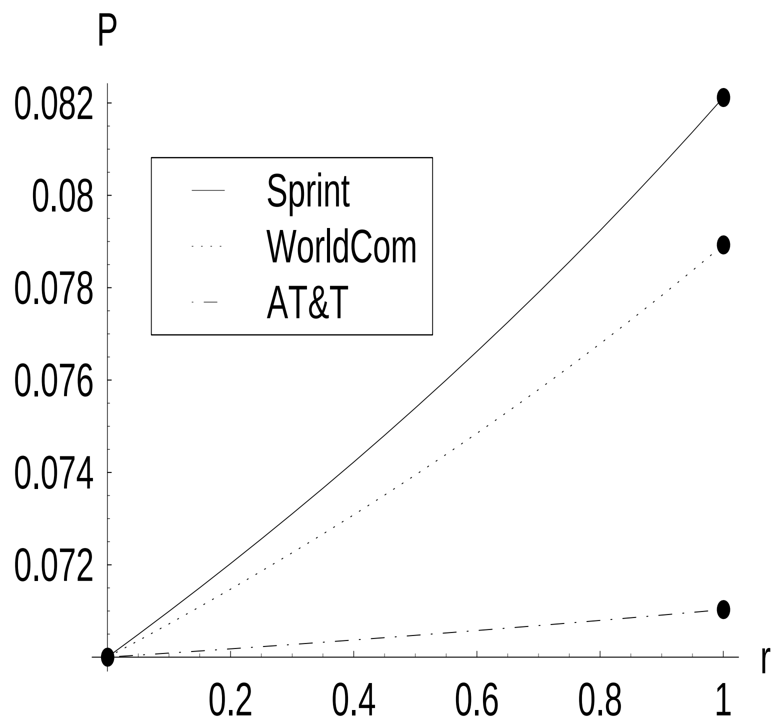

Figure 4: Merger Homotopy Path with AIDS Demand