Mergers Among Asymmetric Bidders:
A Logit Second-Price Auction Model

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Abstract

In this paper, we adapt the logit qualitative-choice model for use in private-value, asymmetric, second-price auctions. The same properties that have made the logit such an attractive model in qualitative-choice settings, its analytic and numeric tractability and its ease-of-estimation, carry over to auction settings. We develop several different estimators to recover the bidders’ joint value distribution from either aggregate or individual auction data. The model is well-suited to the problem of merger prediction where economists must work with available data under time constraints imposed by the merger statutes. From moment restrictions on winning bids and winning probabilities, we derive a Herfindahl-like formula to predict the effects of mergers.

Keywords: merger; auction; logit.

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1 Introduction

In a private-values Vickrey [25] auction, the price, or winning bid, is determined by the value of the second-place bidder. A merger, or bidding coalition, has the potential to change the identity of the second-place bidder, and thus change the winning price. This occurs only when the merged coalition includes both the winning bidder and the second-place bidder. The frequency of this event and the magnitude of the resulting price change determine the expected merger effect.

Since the expected merger effect is obviously dependent on the characteristics of the bidders' joint value distribution, estimation of the value distribution is a crucial step in analyzing the competitive effects of mergers. This paper is motivated by the problems of merger prediction and estimation of the value distribution from observed bidding data.

To study mergers, we need an asymmetric model because, even if bidders are symmetric before the merger, after the merger they are not. In such asymmetric auctions, expected prices and winning probabilities are characterized by multi-dimensional integrals. The difficulty of working with these integrals has made it difficult for economists to develop structural estimators of asymmetric auctions (for Bayesian approach see Bajari [2]). The problem is analogous to the econometric problem of estimating choice probabilities in random-utility models (e.g. Geweke [14]). The analogy to random-utility models suggests also suggests adapting the logit model, with its tractable closed-form expressions, for use in second-price auctions.

Though restrictive (Hausman and McFadden [16]), the logit qualitative-choice model has a number of desirable properties.

- Its analytic tractability makes it an attractive theoretical benchmark. The logit model has facilitated analysis of differentiated-products oligopoly in general (Anderson, DePalma and Thisse [1]), and mergers in particular (Willig [29] and Werden and Froeb [27]).
- The closed-form expressions facilitate estimation with either grouped or individual data (e.g. Train [22]).
- The logit model can be generalized by adding “nests” (McFadden [20]) or “mixing” terms to accommodate consumer heterogeneity (Berry, Levinsohn, and Pakes [6]; Brownstone and Train [8]).

In this paper, we show that the first two of these properties carry over to the logit auction model. In particular we derive closed-form expressions for
the winning probabilities and winning bids in a second-price auction. We conjecture that these closed form expressions can be modified to accommodate nests or mixing terms.

2 A Logit Second-Price Auction Model

Without loss of generality, we assume a selling auction where the high-value firm outbids all the other firms, and where price is set by the second-highest value. We make the assumption that firms’ values are the sum of two independent components: an idiosyncratic component, $X_i$, and a common component with mean zero and finite variance, $Y$. The common component is added to account for between-auction heterogeneity. Thus we let

$$V_i = X_i + Y$$  \hspace{1cm} (2.1)$$

where $V_i$ is the value drawn by the $i$-th bidder, $X_i$ is an independent extreme-value process with parameters $(\eta, \mu)$, and $Y$ is an independent random variable with mean zero and variance $\sigma^2$.

The common component, $Y$, is normalized to have mean zero by adjusting the means of the idiosyncratic components. We assume that the common component is common knowledge; and in much of what follows we ignore the common component because it does not change the winner of the auction or the expected winning bid. It does, however, mask variation in $X_i$ which makes it difficult for estimators to recover the idiosyncratic variance.

The extreme-value distribution, also known as the Gumbel or Type III extreme-value, has the probability density function

$$f(\tau) = \mu e^{-\mu(\tau - \eta)} e^{-e^{-\mu(\tau - \eta)}}$$  \hspace{1cm} (2.2)$$

and the cumulative distribution function

$$F(t) = \int_{-\infty}^{t} f(\tau) d\tau = e^{-e^{-\mu(t - \eta)}}$$  \hspace{1cm} (2.3)$$

The distribution is characterized by location and spread parameters, $(\eta, \mu)$, which are related to its first two moments as follows

$$E(X) = \eta + \frac{\gamma}{\mu}$$  \hspace{1cm} (2.4)$$

\[ \text{Var}(X) = \frac{\pi^2}{6\mu^2} \]  

(2.5)

The symbol \( \gamma \) denotes Euler’s constant (\( \approx 0.57721 \)). The extreme-value distribution is preserved over linear transformations, i.e., if \( X \) is distributed as an extreme-value with parameters \((\eta, \mu)\) then \((X - \eta)\mu\) is distributed as an extreme-value with parameters \((0, 1)\).

The usefulness of the extreme-value distribution for modeling auctions is derived from its closure under the maximum function. If bidders are drawing from independent extreme-value distributions with the same variance, but different means, then the maximum of their values has an extreme-value distribution with the same variance, but a higher mean. The maximum function is used to compute the winning probabilities (the probability that a bidder will have a value higher than the maximum of rivals’ values) and prices (the maximum of rivals’ values), and to compute the effects of a merger (the merged firm has a value equal to the maximum of coalition member values). The following proposition states a well-known property of independent extreme-value distributed variates (see e.g. Ben-Akiva and Lerman [5]).

**Proposition 1.** If \( X_1, X_2, \ldots, X_n \) are distributed as independent extreme value variates with parameters \((\eta, \mu)\), then \(X_{\text{max}} = \max\{X_i : 1 \leq i \leq n\} \) is also distributed as an extreme-value variate with parameters \((\eta_{\text{max}}, \mu)\) where

\[ \eta_{\text{max}} = \frac{1}{\mu} \log \left( \sum_{j=1}^{n} e^{\mu \eta_j} \right). \]  

(2.6)

The next proposition is a standard result for models using the extreme-value distribution. It provides a formula for the probability that bidder \( i \) wins the auction.

**Proposition 2.** Let \( X_{\sim i} = \max\{X_j : 1 \leq j \leq n, j \neq i\} \), and let \( p_i \) be the expected share of bidder \( i \), i.e., the probability that \( X_i > X_{\sim i} \). Then

\[ p_i \equiv \text{Prob}(X_i > X_{\sim i}) = \frac{e^{\mu \eta_i}}{e^{\mu \eta_{\text{max}}}} = \frac{e^{\mu \eta_i}}{\sum_{j=1}^{n} e^{\mu \eta_j}}. \]  

(2.7)

To establish this proposition, we first prove a lemma that is used in the proofs of this and a following proposition.
Lemma 1. The probability that \( X_i \) is the highest value \( X_{\text{max}} \), given that \( X_{\text{max}} \) is less than \( t \), is independent of \( t \), and has value

\[
\text{Prob}(X_i > X_{\text{max}} | X_{\text{max}} < t) = \frac{e^{\mu_{ij}}}{e^{\mu_{	ext{max}}}}.
\]

Proof. See Appendix.

Proposition 2 is an immediate consequence of Lemma 1 by taking the limit as \( t \to \infty \).

Note that the probability that bidder \( i \) has the maximum value is independent of the value of the maximum. This property simplifies derivation of the probability of observing a particular ordering among the top bidders.

Proposition 3. Let \( X_{\text{max}} = \max\{X_m : 1 \leq m \leq n, m \neq i, j\} \) and let \( p_{\text{max}} = \text{Prob}(X_{\text{max}} > \max\{X_i, X_j\}) \). Then

\[
\text{Prob}(X_i > X_j > X_{\text{max}}) = \text{Prob}(X_i > X_{\text{max}}) \cdot \text{Prob}(X_j > X_{\text{max}}) \\
= \left( \frac{p_i}{p_i + p_j + p_{\text{max}}} \right) \left( \frac{p_j}{p_j + p_{\text{max}}} \right) \\
= \left( \frac{p_i}{1 - p_i} \right) p_j
\]

Proof. See Beggs, Cardell, and Hausman [4], and note that

\[
\sum_{j=1}^{r} e^{\mu_{ij}} = \sum_{j=1}^{r} p_j
\]

and \( p_i + p_j + p_{\text{max}} = 1 \).

The formula above can be extended in a straightforward manner to derive the probability of observing any ordering among the top \( r \) of \( n \) bidders where \( r \leq n \).

Next we examine the distribution of winning bids for each auction participant. The general formulas for order statistics drawn from heterogeneous distributions can be found in Section 2.8 of David [10]. What differentiates our approach from the development in [10] is that we compute order statistics for the second-highest value, conditional on the identity of the high-value bidder. By conditioning on the identity of the bidders, we are implicitly assuming that the researcher has data on bidder identities.
Proposition 4. Let $A_i$ be the highest value of the $X_i$'s conditional on bidder $i$ being the winning bidder, i.e., $A_i$ is $X_i$ conditional on $X_i > X_{\infty i}$. Then $A_i$ is distributed identically to $X_{\text{max}}$ (the same for any $i$). Let $B_i$ be the second-highest value of the $X_i$'s conditional on bidder $i$ being the winning bidder, i.e., $B_i$ is $X_{\infty i}$ conditional on $X_{\infty i} < X_i$. Then the cumulative distribution function of $B_i$ is

$$F_{B_i}(t) = \left(\frac{1}{p_i}\right) F_{X_{\infty i}}(t) + \left(1 - \frac{1}{p_i}\right) F_{X_{\text{max}}}(t). \quad (2.8)$$

Proof. See Appendix. $\square$

Equation 2.8 can be used to compute the mean and variance of the $i$-th bidder’s winning bid. We extend the formulas to the case where we include an common shock to all bidders’ values $Y$ with mean 0 and variance $\sigma^2$

$$E(B_i + Y) = E(X_{\text{max}}) + \frac{\log(1 - p_i)}{\mu p_i} \quad (2.9)$$

$$\text{Var}(B_i + Y) = \sigma^2 + \frac{\pi^2}{6\mu^2} \left[1 - \frac{6}{\pi^2} (\log(1 - p_i))^2 \left(\frac{1}{p_i^2}\right)\right]. \quad (2.10)$$

In Equation 2.9, the expected price is decreasing in the expected share because a high-mean-value firm doesn’t bid against itself, making it more likely that it will win at a lower price than a low-mean-value firm.

It is interesting to note the parallels of this auction model to other asymmetric oligopoly models. They typically exhibit positive margin/share relationships, similar to Equation 2.9. In the asymmetric Cournot model, the margin/share relationship is implied by the Nash first-order conditions

$$\frac{\text{price} - mc_i}{\text{price}} = \frac{\text{share}_i}{\epsilon} \quad (2.11)$$

where $\epsilon$ is the price elasticity of demand. In the Bertrand model, the Nash first-order conditions are written in terms of a margin/elasticity relationship

$$\frac{\text{price}_i - mc_i}{\text{price}_i} = \frac{1}{\epsilon_i} \quad (2.12)$$
Usually, elasticity is inversely related to share (large shares are associated with lower elasticities), as in the logit differentiated products model (e.g. Werden and Froeb [27]). A negative elasticity/share relation implies a positive margin/share relationship. Note that these relationships rule out the existence of high-volume, low-margin firms in Nash equilibrium.

3 Estimating the Value Distribution

Auctions are typically used to sell or purchase unique items. Because of this, auction data are characterized by a considerable degree of heterogeneity across items. We have modeled this heterogeneity by adding a common observed component, $Y$, to each bidder’s value. This common component masks variation in winning bids due to the idiosyncratic component and implies that estimators based only on the winning bids will have a difficult time distinguishing within from between-auction variance. This leads us to consideration of a within-auction estimator, based on differences between losing bids.

In what follows, we assume the existence of data on bidder identities, bidder characteristics (including losing bidder characteristics), and winning bids across a sample of auctions. We treat each auction as an independent event, but recognize that this assumption may not be appropriate in the presence of collusion, as in a bid-rotation scheme, or with bidder capacity constraints.

3.1 Individual Data: Winning Bids

In this section, we construct a two-step methods-of-moments estimator of the value distribution using data on winning bids and bidder characteristics for a set of auctions. In the first step, winning bidders are “predicted” as a function of observable bidder characteristics using a maximum-likelihood logit estimator. The logarithm of the likelihood function is constructed from the probability of winning among $T$ bidders as

$$L = \sum_{t=1}^{T} \log(p_{it}) = \sum_{t=1}^{T} \log \left( \frac{e^{\eta_{it}}}{\sum_{k=1}^{n_t} e^{\eta_{kt}}} \right)$$

(3.1)

the variable $i$ is taken as a function of $t$ that gives the winning bidder in the $t$-th auction, i.e. $p_{it}$ is the probability that the $t$-th auction is won by bidder $i$. The estimated location parameters, $\hat{\eta}_{it}$, are the fitted values from
the logit estimation. We note that the estimated location parameters are observationally equivalent to parameters scaled by a linear transformation, \( \text{i.e.} \), the logit probabilities for an extreme value \((\tilde{\eta}_u, 1)\) are identical to those for an extreme value \(((\tilde{\eta}_u + c)/\mu, \mu)\).

The linear scaling, \( \text{i.e.} \) the parameters \(c \) and \( \mu \), are identified by substituting the scaled location parameters into Equation 2.9 to express the first moment restriction on the \(t\)-th auction winning bid in terms of the two unknowns, \((c/\mu, 1/\mu)\). Namely,

\[
E(b_{it}) = E(Y_t) + \frac{c}{\mu} + \frac{1}{\mu} \left( \gamma + \tilde{\eta}_{max} + \frac{\log(1 - \tilde{p}_it)}{\tilde{p}_it} \right)
\]  

where \(b_{it}\) is the observed winning bid, \( \text{i.e.} \) the realized values of the random variable, \(B_{it}+Y_t\). Equation 3.2 may be estimated using a minimum distance estimator to recover the unknown parameters, \((c/\mu, 1/\mu)\). This is equivalent to a regression of the winning bids on a constant and the term in brackets in Equation 3.2. The expected value of the common value component, \(E(Y_t)\) in Equation 3.2, can be specified as a function of observable auction characteristics. With this procedure, the parameters of the individual value distributions are estimated as \((\tilde{\eta}_u + \hat{c})/\hat{\mu}\).

### 3.2 Aggregate Data: Winning Bids

Sometimes, especially in merger investigations, only aggregated data, \( \text{i.e.} \) data grouped by bidder, are available. With aggregate data on prices (average winning bids) and shares (how frequently bidders' win), it is possible to use the same two-step estimator as above. However, with aggregate data it is also possible to use a maximum-likelihood estimator by treating the aggregate prices as means and applying the central limit theorem. Let \(\bar{p}_i\) and \(\bar{b}_i\) denote the share and the average winning bid for bidder \(i\) across a sample of \(T\) auctions. The joint density of the share and the winning bid can be factored \( \text{i.e.} \), \(f(\bar{b}_i, \bar{p}_i) = f(\bar{p}_i) \cdot f(\bar{b}_i|\bar{p}_i)\), so that log-likelihood is the sum of two components. The first component is the logit log-likelihood, and the second component is the log-likelihood for a normal distribution,

\[
L = \sum_{i=1}^{k} T_i \log(p_i) + \sum_{i=1}^{k} \left[ -\frac{\log(2\pi)}{2} - \frac{\log(Var(\bar{b}_i))}{2} - \frac{(\bar{b}_i - E[\bar{b}_i])^2}{2Var(\bar{b}_i)} \right].
\]  

Here \(k\) is the number of bidders; \(T_i\) is the number of auctions won by bidder \(i\); and \(Var(\bar{b}_i) = Var(B_{i}+Y)/T_i\) and \(E[\bar{b}_i] = E[B_{i}+Y]\) are defined by Equations 2.9 and 2.10, respectively.
3.3 Individual Data: Within-Auction Estimators

Estimators based only on winning bids have two problems. First, they have difficulty distinguishing a large idiosyncratic variance (small $\mu$), from a large common variance (large $\sigma$). The difficulty can be understood by examining the method-of-moments estimator. The variance parameter is estimated from the slope of the price/share relationship. Large price variance, due either to common or idiosyncratic shocks, is analogous to a large residual variance, which reduces the precision of the slope estimator.

The second problem is that bidder participation in the auction may be correlated with an unobservable (to the econometrician) element of the common shock, $Y$. This leads to simultaneous equations bias caused by the endogeneity of the bidder pool. This is analogous to the endogeneity of concentration in a price/concentration regression (Evans, Froeb, and Werden [12]). We address both concerns with within-auction estimators based on the difference between losing bids. Differentiating the bids eliminates the nuisance variable $Y$.

3.3.1 Second-price Auctions

In a Vickrey [25] or second-price auction, the highest bidder pays a price equal to the second-highest bid. Because the winning bidder’s price is not dependent on his bid, it is a dominant strategy for each bidder to bid his true value. A maximum-likelihood estimator can be constructed from the distribution of the difference between the highest and second-highest bids. Note that this is also the surplus or profit margin to the winning bidder.

**Proposition 5.** Let $\Delta_i = A_i - B_i$ be the difference between the highest and second-highest values, conditional on bidder $i$ having the highest value. The cumulative distribution function of $\Delta_i$ is

$$F_{\Delta_i}(t) = 1 - \left( \frac{1}{p_i + p_{-i}e^{\text{id}}} \right).$$  \hspace{1cm} (3.4)

**Proof.** See Appendix.

Expressing the likelihood in terms of the $p_i$'s instead of the $\eta_i$'s suggests a two-step limited-information maximum-likelihood estimator analogous to the two-step method-of-moments estimator above. In the first step, logit probabilities are estimated with a maximum-likelihood routine as in Equation 3.1. The fitted probabilities are used to construct the likelihood based on
the density corresponding to the distribution $F_{A_i}(t)$ of Proposition 5. The parameter $\mu$ is recovered in the second-stage estimation. The corresponding full-information maximum-likelihood estimator can be estimated using starting values obtained from the limited-information estimator. In our experience, convergence is more likely if the likelihood is parameterized in terms of $1/\mu$ rather than $\mu$.

### 3.3.2 Open Auctions

For certain open auctions, it is possible to observe losing bids. For example, in government procurement, losing oral bids are recorded (e.g., Brannman and Froeb [7]). Depending on the precise bidding mechanism, the difference between the second and third-highest bids can be taken as the difference between the second and third-highest values because it is a dominant strategy for losing bidders to bid up to their values. In contrast, the difference between the two highest bids is not informative about the distribution because the winner is trying only to outbid the second highest-value bidder. Information about the differences between lower-ranked bids may be less informative for the same reasons that information about lower-ranked surveyed alternatives is less precise (Hausman and Rund [17]).

**Proposition 6.** As above, let $B_i$ be the second-highest value of the $X_i$'s after $X_i$. Let $B_{i,j}$ be the third-highest value of the $X_i$'s behind $X_i$ and $X_j$, in that order. Let $\Delta_{i,j} = B_i - B_{i,j}$ be the difference between the second and third highest values. The cumulative distribution function of $\Delta_{i,j}$ is

$$F_{\Delta_{i,j}}(t) = 1 - \left( \frac{p_j + p_{i,j}}{p_j + p_{i,j}e^{\mu t}} \right) \left( \frac{p_i + p_j + p_{i,j}}{p_i + p_j + p_{i,j}e^{\mu t}} \right). \quad (3.5)$$

**Proof.** See Appendix. \qed

The limited and full-information maximum-likelihood estimators from Section 3.3.1 are available to estimate this equation.

### 4 Predicting Merger Price Effects

U.S. antitrust law prohibits mergers that substantially lessen competition. Once consummated, mergers are very costly to undo, so predictions about merger effects must be made prior to consummation. These predictions are made in an adversarial setting, and under the time constraints mandated
the merger statutes. To be useful for policy, merger predictors must be “objective and tractable” (Sherwin [21]) and flexible enough to be used with available data.

Currently, the courts and enforcement agencies use “structural” predictors to evaluate proposed mergers. Markets are delineated, shares are assigned, and the legality of a merger turns largely on the size of the merging firms’ shares. Structural predictors do not perform well because they are not based on economic models of oligopoly, on empirical studies of mergers, or on empirical studies of the structure/performance relationship (Werden and Froeb [28]). In this section, we propose an alternative to structural merger analysis by applying the logit auction model to the problem of predicting merger price effects.

Although mergers are typically very complex transactions, their direct anticompetitive effect in the private-value auction setting can be modeled by assuming that the merged entity generates values equal to the maximum of the pre-merger individual values. Like mergers among producers of differentiated products (Davidson and Deneckere [11]), such mergers are always profitable (Mailith and Zemsky [18]) because the merged entity has the same stand-alone value as its coalition members, and it eliminates competition among them. This merger characterization has been used by the antitrust enforcement agencies to model the price effects of mergers in various auction settings, including mining equipment and hospitals (Baker [3]).

In the logit auction model, Equation 2.9 can be used to compute expected profits for each bidder, the surplus of value over winning bid for the fraction of auctions expected to be won. The expected profit to bidder $i$ is thus measured by

$$[E(X_{max}) - E(B_i)] p_i = - (1/\mu) \log (1 - p_i). \quad (4.1)$$

Let $h(x) = - \log(1 - x)$ so that the profit for bidder $i$ is $h(p_i)/\mu$. Then the total profit to all bidders is

$$H = \frac{1}{\mu} \sum_{i=1}^{n} h(p_i). \quad (4.2)$$

The expression for $H$ is analogous to a Herfindahl index, and can be used to compute the expected change in industry profits following a merger between firms $i$ and $j$. The share of the merged firm is $p_i + p_j$ since the merged firm

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wins in every auction where one or the other of the original firms would have won. The shares of the other firms remain unchanged. Since the auction is efficient, the extra profit to firms $i$ and $j$ is equal to the reduction in revenue at auction, and is given by

$$\Delta H = \Delta Revenue = \frac{h(p_i + p_j) - h(p_i) - h(p_j)}{\mu},$$

(4.3)
a simple function of the shares of the merging firms scaled by the spread parameter.

The logit auction model can also be used to construct a consumer-welfare benchmark, against which the efficiency claims of the merging parties can be evaluated. Although mergers do not affect the efficiency properties of second-price auctions, they do harm consumers (i.e., the auctioneer) by reducing the winning bids of the merging firms. Merger synergies can offset this harm by making the merged firm a stronger bidder. This forces the nonmerging firms to bid higher in order to win. In this way, mergers can raise prices paid by non-merging firms just as they lower prices paid by merging firms. The increase in value (reduction in the marginal costs) of the merged firm that keeps industry price constant can be numerically computed by equating pre and post-merger expected price.

For large mergers, such compensating marginal cost reductions may not exist. Once the merging firm reaches a certain size, no amount of marginal cost reduction can offset the merger price increase because marginal cost reductions increase price only in auctions that the merging firms do not win. If the merging firms are already winning most of the auctions, then making them stronger cannot offset the merger price effects.

5 Discussion

In the model, we find that the effects of mergers are critically dependent on the variance of the bidders' joint value distribution. Depending on the variance, a highly concentrated industry structure could be consistent with grossly different price effects, at least some of which would not be considered anticompetitive (or vice-versa). A similar finding, that the effects of merger are critically dependent on the structure of demand (Hausman, Leonard, and Zona [15]; Werden and Froeb [27]) has lead to criticism of structural merger policy in differentiated products industries. We have shown that the same criticism applies to structural merger policy in auction markets.
Estimating the bidders’ joint value distribution is just as important for evaluating mergers in auction markets as estimating demand is for evaluating mergers in differentiated products industries.

6 Appendix: Proofs of Lemma and Propositions

Lemma 1. The probability that $X_i$ is the highest value $X_{\text{max}}$, given that $X_{\text{max}}$ is less than $t$, is independent of $t$, and has value

$$
\text{Prob}(X_{\infty} < X_i | X_{\text{max}} < t) = \frac{e^{\eta_i}}{e^{\theta \eta_{\text{max}}}}.
$$

Proof. For $\Delta t > 0$, we have

$$
\text{Prob}(X_{\infty} < X_i | X_{\text{max}} \in [t, t+\Delta t]) = \frac{\text{Prob}(X_{\infty} < X_i \land X_{\text{max}} \in [t, t+\Delta t])}{\text{Prob}(X_{\text{max}} \in [t, t+\Delta t])}.
$$

We note that if $\Delta t \ll 1$, then

$$
\text{Prob}(X_{\text{max}} \in [t, t+\Delta t]) \approx \frac{dF_{\text{max}}(t)}{dt} = F'_{\text{max}}(t),
$$

where $F_{\text{max}}$ is the cumulative distribution function of $X_{\text{max}}$. In the numerator, when $X_{\infty} < X_i$, we have $X_{\text{max}} = X_i$, and hence, $X_i \in [t, t+\Delta t]$ and $X_{\infty}$ is bounded above by a number between $t$ and $t+\Delta t$. As $\Delta t \to 0$, the probability that $X_{\infty} < t + \Delta t$ approaches the probability that $X_{\infty} < t$. Hence, we have the approximation for $\Delta t \ll 1$,

$$
\frac{\text{Prob}(X_{\infty} < t \land X_{\text{max}} \in [t, t+\Delta t])}{\Delta t} \approx \text{Prob}(X_{\infty} < t) \frac{\text{Prob}(X_i \in [t, t+\Delta t])}{\Delta t} \approx F_{\infty}(t) F'_i(t)
$$

where $F_i$ and $F_{\infty}$ are the cumulative distribution functions of $X_i$ and $X_{\infty}$.

Taking the limit as $\Delta t \to 0$, we have

$$
\lim_{\Delta t \to 0} \text{Prob}(X_{\infty} < X_i | X_{\text{max}} \in [t, t+\Delta t]) = \frac{F_{\infty}(t) F'_i(t)}{F'_{\text{max}}(t)}
$$

$$
= \frac{\mu F_{\infty}(t) F'_i(t) e^{-\theta (t-\eta_i)}}{\mu F_{\text{max}}(t) e^{-\theta (t-\eta_{\text{max}})}}
$$

$$
= \frac{e^{\eta_i}}{e^{\theta \eta_{\text{max}}}}.
$$
since $F_i(t)F_{\ast i}(t) = F_{max}(t)$. But this limit is independent of $t$, and hence, the probability that $X_i$ is largest, given that $X_{max}$ lies in any sufficiently small interval is close to the same constant value. We conclude that the probability is equal to this constant on any interval and so

$$Prob(X_{\ast i} < X_i|X_{max} < t) = e^{\mu_i}/e^{\mu_{max}}$$

is independent of $t$.

**Proposition 4** Let $A_i$ be the highest value of the $X$'s conditional on bidder $i$ being the winning bidder, i.e., $A_i$ is $X_i$ conditional on $X_{\ast i} < X_i$. Then $A_i$ is distributed identically to $X_{max}$ (the same for any $i$). Let $B_i$ be the second-highest value of the $X$'s conditional on bidder $i$ being the winning bidder, i.e., $B_i$ is $X_{\ast i}$ conditional on $X_{\ast i} < X_i$. Then the cumulative distribution function of $B_i$ is

$$F_{B_i}(t) = \left(\frac{1}{p_i}\right) F_{\ast i}(t) + \left(1 - \frac{1}{p_i}\right) F_{max}(t). \quad \text{(6.1)}$$

**Proof.** We consider the cumulative distribution functions of highest and second-highest bids given a particular $X_i$ is largest. Consider the cumulative distribution function for $A_i$, $Prob(X_i < t|X_{\ast i} < X_i)$. We compute

$$Prob(X_i < t|X_{\ast i} < X_i) = \frac{Prob(X_{\ast i} < X_i < t)}{Prob(X_{\ast i} < X_i)} = \frac{Prob(X_{\ast i} < X_i|X_{max} < t)Prob(X_{max} < t)}{Prob(X_{\ast i} < X_i)} = \frac{p_i F_{max}(t)}{p_i} = F_{max}(t)$$

so $A_i$ is distributed identically to $X_{max}$, and is independent of $i$. Next consider the cumulative distribution function for $B_i$, $Prob(X_{\ast i} < t|X_{\ast i} < X_i)$. We now compute
\[ \text{Prob}(X_{\infty i} < t | X_{\infty i} < X_i) \]
\[ = \frac{\text{Prob}(X_{\infty i} < t \text{ and } X_{\infty i} < X_i)}{\text{Prob}(X_{\infty i} < X_i)} \]
\[ = \frac{\text{Prob}(X_{\infty i} < t) - \text{Prob}(X_i < X_{\infty i} < t)}{\text{Prob}(X_{\infty i} < X_i)} \]
\[ = \frac{\text{Prob}(X_{\infty i} < t) - \text{Prob}(X_i < X_{\infty i} | X_{\max} < t) \text{Prob}(X_{\max} < t)}{\text{Prob}(X_{\infty i} < X_i)} \]
\[ = \frac{F_{\infty i}(t) - (1 - p_i) F_{\max}(t)}{p_i} \]

which is the claimed cumulative distribution function for \( B_i \). \( \square \)

**Proposition 5**

Let \( \Delta_i = A_i - B_i \) be the difference between the highest and second-highest values given that bidder \( i \) wins the auction. The cumulative distribution function of \( \Delta_i \) is

\[ F_{\Delta i}(t) = 1 - \left( \frac{1}{p_i + p_{\infty i} \epsilon_{id}} \right). \]  \hspace{1cm} (6.2)

**Proof.** Using Proposition 3 and the observation that \( X + t \) has an extreme-value distribution with parameters \( (\eta + t, \mu) \) if \( X \) has an extreme-value distribution with parameters \( (\eta, \mu) \), we compute

\[ F_{\Delta i}(t) = \text{Prob}(X_i - X_{\infty i} < t | X_i > X_{\infty j}) \]
\[ = \frac{\text{Prob}(X_i - X_{\infty i} < t \wedge X_i > X_{\infty j})}{\text{Prob}(X_i > X_{\infty j})} \]
\[ = \frac{\text{Prob}(X_i > X_{\infty i}) - \text{Prob}(X_i > X_{\infty i} + t)}{\text{Prob}(X_i > X_{\infty i})} \]
\[ = 1 - \frac{\text{Prob}(X_i > X_{\infty i} + t)}{\text{Prob}(X_i > X_{\infty i})} \]
\[ = 1 - \left( \frac{1}{p_i + p_{\infty i} \epsilon_{id}} \right). \]

\( \square \)

**Proposition 6** As above, let \( B_i \) be the second-highest value of the \( X \)'s after \( X_i \). Let \( B_{i,j} \) be the third-highest value of the \( X \)'s behind \( X_i \) and \( X_j \), in that
order. Let $\Delta_{i,j} = B_i - B_{i,j}$ be the difference between the second and third highest values. The cumulative distribution function of $\Delta_{i,j}$ is

$$F_{\Delta_{i,j}}(t) = 1 - \left( \frac{p_j + p_{\sim ij}}{p_j + p_{\sim ij}e^{i\theta t}} \right) \left( \frac{p_i + p_j + p_{\sim ij}}{p_i + p_j + p_{\sim ij}e^{i\theta t}} \right). \quad (6.3)$$

Proof.

$$F_{\Delta_{i,j}}(t) = \text{Prob}(X_j - X_{\sim ij} < t|X_i > X_j > X_{\sim ij})$$

$$= \frac{\text{Prob}(X_j - X_{\sim ij} < t \land X_i > X_j > X_{\sim ij})}{\text{Prob}(X_i > X_j > X_{\sim ij})}$$

$$= \frac{\text{Prob}(X_i > X_j > X_{\sim ij}) - \text{Prob}(X_i > X_j > X_{\sim ij} + t)}{\text{Prob}(X_i > X_j > X_{\sim ij})}$$

$$= 1 - \frac{\text{Prob}(X_i > X_j > X_{\sim ij} + t)}{\text{Prob}(X_i > X_j > X_{\sim ij})}$$

$$= 1 - \left( \frac{p_j + p_{\sim ij}}{p_j + p_{\sim ij}e^{i\theta t}} \right) \left( \frac{p_i + p_j + p_{\sim ij}}{p_i + p_j + p_{\sim ij}e^{i\theta t}} \right)$$

$$= 1 - \left( \frac{p_j + p_{\sim ij}}{p_j + p_{\sim ij}e^{i\theta t}} \right) \left( \frac{1}{p_i + p_j + p_{\sim ij}e^{i\theta t}} \right) \quad \square$$
References


