Modeling implications of source-invariance to Machina's 'almost objective fair bets' *

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Abstract

Machina (2004) introduced the notion of an 'almost objective' event in a continuous state space—high frequency events in a subjective setting such as 'the realization of the n^{th} decimal place of a stock index.' Payoffs on such events intuitively appear as objective lotteries in the sense that decision makers should not prefer to place bets on any particular digit when n is large even if the state space is fully subjective. This paper investigates the implications of requiring decision makers to treat almost objective events the same regardless of their source (e.g., regardless of the identity of the stock index). Multi-prior models in which the set of representing priors are smooth (i.e., possess densities) can accommodate such source indifference. The major contribution of this paper is to demonstrate that, under mild behavioral conditions, a multi-prior representation with smooth priors is also necessary.

Keywords: Uncertainty, Risk, Ambiguity, Decision Theory, Non-Expected Utility, Multiple Priors.

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1 Introduction

Consider an investor who contemplates allocating her funds between stock portfolios mimicking the Japanese Nikkei index and/or the American Dow Jones Industrial Average (DJIA). Suppose one asks this investor the following questions before she makes her investment decision:

Q1: At what level of the Nikkei index, l_N, would you be indifferent to betting that next week the Nikkei will be above versus below l_N?
Q2: At what level of the DJIA, l_D, would you be indifferent to betting that next week the DJIA will be above versus below l_D?

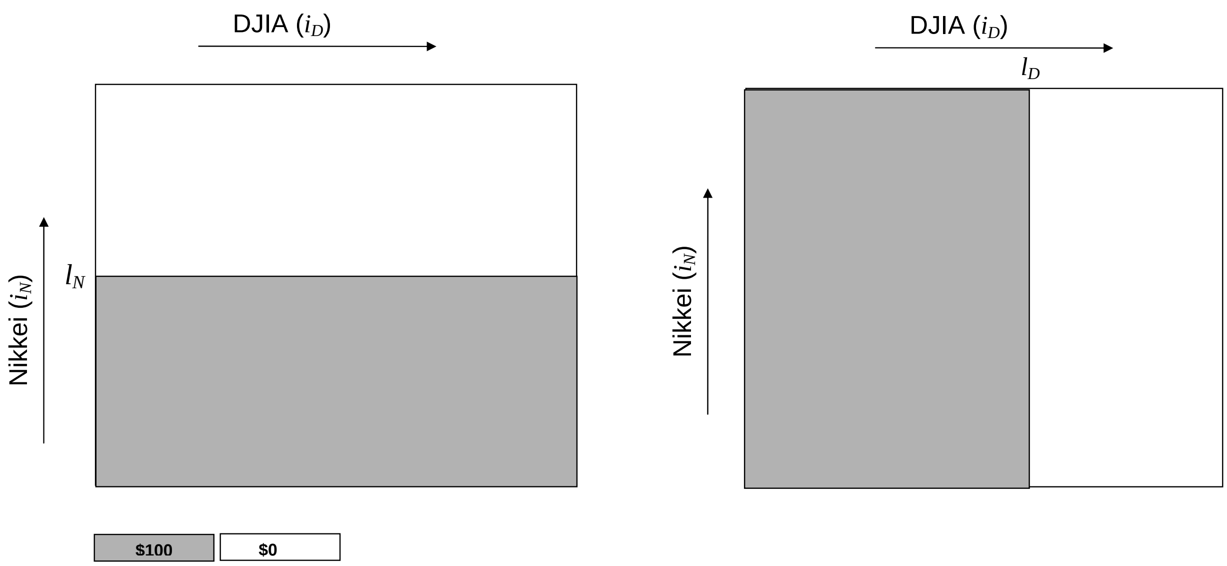

Figure 1: The decision maker is asked to select a subjective median for the Nikkei (l_N on the left) and for the DJIA (l_D on the right). Although she is indifferent to betting \$0 or \$100 on either side of the subjective median, the decision maker might strictly prefer to bet on a partition of the Nikkei, consistent with the Ellsberg Paradox.

Suppose, for the sake of exposition, that these bets are unrelated to the decision maker's uninvested wealth, so that there is no hedging motive to consider. One might therefore imagine that her answers reflect her best estimate of the median forecasts for each financial indicator (see Figure 1). At the same time, this investor, being Japanese, might exhibit a preference for betting above l_N on the Nikkei rather than above l_D on the DJIA (perhaps because she is ambiguity averse, or has 'source-dependent' preferences).[1] If one wishes to model the investor's Knightian attitudes toward investment in Japan and the United States, an intuitive, if simplistic, model might assess the return trade-offs as follows. Let (i_D, i_N) be a possible joint realization of the DJIA and Nikkei, respectively, in exactly one week. Let $\rho(i_D, i_N)$ be some joint subjective probability density over these realizations, and let $\rho_D(i_D)$ and $\rho_N(i_N)$ be the associated marginal densities. Then, consider the following certainty equivalent to a bet that pays $f(i_D, i_N)$ in the event that $(i_D, i_N) \in [i_D, i_D+di_D]\times[i_N, i_N+di_N]$:

$$\text{ce}(f) = \bar{f} - A\sigma(f) - a\int \sigma_{D,f}(i_N)\ \rho_N(i_N)di_N, \tag{1}$$

where A and a are positive constants, $\bar{f}$ and $\sigma(f)^2$ are the mean and variance, respectively, of the payoff profile f as calculated according to $\rho(i_D, i_N)$, while $\sigma_{D,f}(i_N)^2$ is the variance of these payoffs conditional on i_N. The first two terms on the right side of Eq. (1) represent standard mean-variance preference, calculated using the subjective density, $\rho(i_D, i_N)$. The third term represents additional aversion to uncertainty that arises from bets in which the payoffs vary over DJIA realizations. Specifically, the last term calculates the standard deviation of the act conditional on each Nikkei realization, and then aggregates the results. Assuming $\rho(i_D, i_N) = \rho_D(i_D)\rho_N(i_N)$, the bet on the Nikkei in Figure 1 yields a certainty equivalent of $50(1-A)$ because there is no uncertainty along the DJIA direction. By contrast, the bet on the DJIA in the same figure yields $50(1-A-a)$, for a Knightian uncertainty premium proportional to a. Thus the representation in (1) can be used to model the types of uncertainty attitudes demonstrated in the Ellsberg Paradox, or the 'home bias' literature (see French and Poterba, 1991; Huberman, 2001; Coval and Moskowitz, 2001; Uppal and Wang, 2002).

Assume that the same investor is subsequently asked:

Q3: Would you prefer to bet that the n^{th} trailing digit of the Nikkei index next week will

[1] See Wakker (2006) for a literature review on ambiguity aversion and source preference. Models of source preference are developed in Chew and Sagi (2003), Ergin and Gul (2004), Nau (2006), Schroder and Skiadas (2003), and Skiadas (2008).

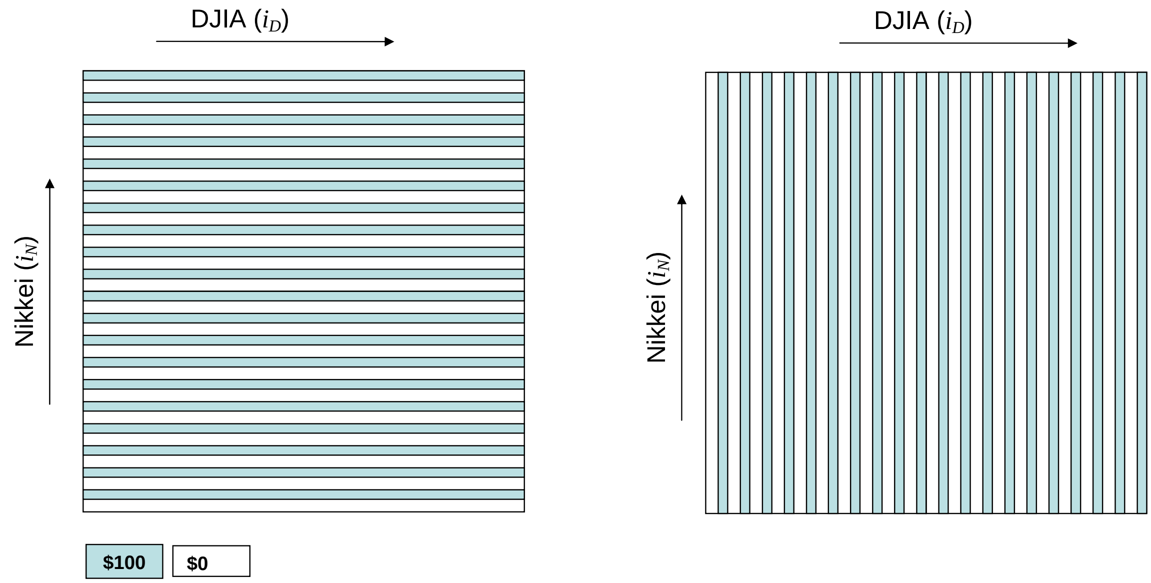

Figure 2: The diagrams depict bets on whether the realization of the n^{th} trailing digit of the Nikkei index (left) or the DJIA (right) is odd or even. As n approaches infinity, each of the bets is treated the same, a 50-50 gamble with payoffs $100 and $0, and the decision maker is indifferent between them.

be even or odd?

Q4: Would you prefer to bet that the n^{th} trailing digit of the DJIA next week will be even or odd?

These types of bets are depicted in Figure 2. If the Nikkei index or DJIA can be reported to arbitrary accuracy, and subject to the absence of superstition or known biases in stock price quotations, our investor would likely be close to indifferent between betting on any of the four partitions if n is greater than, say five (i.e., the certainty equivalents of the bets depicted in Figure 2 should be very close).[2] As n increases it seems sensible to expect or even require the certainty equivalents of these bets to converge despite the potential presence of ambiguity or Knightian uncertainty. In contrast with this behavioral intuition, the certainty

[2]The Nikkei index and DJIA are currently reported to 7 digits. The portfolio composition corresponding to each of these indicators is public knowledge, thus one can in principle calculate them to higher accuracy. Biases in trailing digits can, in practice, exist: Christie and Schultz (1994) found that NASDAQ dealers implicitly colluded to inflate transaction costs by avoiding odd-eighth quotes on individual stocks. Neither the researchers nor the market suspected this, and once discovered the practice was brought to an end.

equivalent specification in (1) predicts that the bet depicted on the left of Figure 2, and corresponding to Q3, will have a certainty equivalent approaching $50(1 - A)$ as $n \to \infty$, while the bet corresponding to Q4 will, in the same limit, approach $50(1 - A - a)$. In other words, if one adopts as behaviorally normative the principle that decision makers ought to be close to indifferent between the type of bets depicted in Figure 2, then a representation like that in (1) must be ruled out.

This paper is motivated by the view that asymptotic indifference to the bets in Q3 and Q4, as n grows large, is both descriptively plausible and behaviorally weak. As exemplified by the ruling out of (1), this principle of asymptotic indifference constrains the modeling of Knightian or source-dependent preferences. It is therefore important to investigate general implications of such a condition.

Machina (2004) introduced the notion of an 'almost objective' event in a Euclidean state space—high frequency events in a subjective setting, like those depicted in Figure 2. Machina observes that in a large class of existing models for decision making, a decision maker (DM) will perceive bets on 'almost objective' events as risky, or purely mechanical, 50:50 gambles. In the models he considers, a bet on whether the n^{th} decimal digit of a future temperature reading will be even or not, as n approaches infinity, has the same certainty equivalent as a bet on a fair coin flip. Specifically, Theorem 8 of Machina (2004) establishes that for 'event-smooth preferences' (see Machina, 2004, 2005), the certainty equivalents of the two bets depicted in Figure 2 asymptotically converge. This paper attempts to reverse the set of questions posed by Machina (2004). Instead of focusing on a class of preferences and asking whether they treat bets on 'almost objective' events as risky gambles, the goal here is to ask:

> Are there broad implications for models of decision making under uncertainty when, in addition to the conventional preference requirements (e.g., completeness, transitivity, continuity, monotonicity, etc.), one requires DMs to treat Machina's 'almost objective fair bets' as coin flips regardless of how they are generated?

The answer provided here is that, under mild behavioral conditions, a multi-prior representation, where the priors satisfy a list of properties, is necessary. Moreover, under certain circumstances, also characterized here, the set of representing priors is unique and has a natural interpretation in terms of an incomplete likelihood relation over events. The remainder of the Introduction elaborates on these findings.

1.1 Smooth multi-prior representations and 'almost objective fair bets'

Consider 'smooth prior distributions' on the space spanned by joint realizations of the DJIA and Nikkei: these are all the probability measures that are absolutely continuous with respect to the Lebesgue measure. I.e., if an event is assigned zero Lebesgue measure, then the smooth prior also assigns it zero measure.[3] Machina (2004, see page 31) conjectures, and it is substantiated in Proposition 1 of this paper, that as n approaches infinity in Q3 and Q4, *any* smooth prior will assess the distribution of each of the bets as 50:50 regardless whether the source is the Nikkei or the DJIA. Thus, smooth prior distributions do not distinguish between sources of 'almost objective' events. This observation suggests that one can model both Knightian attitudes toward uncertainty and impose indifference to the source of 'almost objective' events by using a multi-prior utility representation, where each prior is smooth. To see how this may be done, let a payoff profile, such as one of those depicted in Figures 1 or 2, be denoted as f, and a prior distribution over states be denoted as μ. The prior μ can be used to assess the probability of payoffs implied by f to arrive at a lottery, denoted as $L_{\mu,f}$. Let f_n^{Q3} denote the payoff profile corresponding to Q3 and represented in Figure 2, and similar for f_n^{Q4} (the integer n is the order of the trailing digit in Q3 and Q4). Then, it is shown in Proposition 1, that if μ is smooth, then

$$\lim_{n\to\infty} L_{\mu,f_n^{Q3}} = \lim_{n\to\infty} L_{\mu,f_n^{Q4}} = \text{50:50 chance at \$100 or \$0.} \tag{2}$$

[3]It is important to emphasize that smooth priors, as defined here, are distinct from the notion of event-smooth preferences introduced by Machina (2004, 2005). In particular, the term 'smooth', as used here, only conveys 'differentiability' in the sense that a smooth measure has a Radon-Nikodym derivative with respect to the Lebesgue measure.

A decision maker is said to be probabilistically sophisticated if her preferences over payoff profiles can be represented using a function of the form $v(L_{\mu,f})$ for some prior, μ (see Machina and Schmeidler, 1992; Grant, 1995; Chew and Sagi, 2006). Thus, if μ for such a decision maker happens to be smooth, then in light of Eq. (2), for the decision maker to be asymptotically indifferent between f_n^{Q3} and f_n^{Q4} as n grows large, it is sufficient to require that

$$\lim_{n\to\infty} v\big(L_{\mu,f_n^{Q3}}\big) = v\big(\lim_{n\to\infty} L_{\mu,f_n^{Q3}}\big) \text{ and } \lim_{n\to\infty} v\big(L_{\mu,f_n^{Q4}}\big) = v\big(\lim_{n\to\infty} L_{\mu,f_n^{Q4}}\big).$$

This condition is weaker than Machina's (2004, Theorem 8) requirement that $v(\cdot)$ be event-smooth.

It is now commonplace to model preferences of a decision maker who is not probabilistically sophisticated using a *multi-prior representation*:

$$V(f) = v\big(L_{\mu_1,f}, \ldots, L_{\mu_i,f}, \ldots\big), \tag{3}$$

where each of the μ_i's are distinct prior distributions over states, and i ranges over a potentially uncountable index set.[4] Examples of such utility representations include Gilboa and Schmeidler (1989), Ghirardato and Marinacci (2002), Klibanoff, Marinacci, and Mukerji (2005), or Maccheroni, Marinacci, and Rustichini (2006). If each of the μ_i's are smooth, then for every i, $\lim_{n\to\infty} L_{\mu_i,f_n^{Q3}}$ tends to a 50:50 lottery paying \$100 or zero, and the same is true for $\lim_{n\to\infty} L_{\mu_i,f_n^{Q4}}$. Thus, in analogy with the probabilistically sophisticated decision maker, asymptotic indifference between f_n^{Q3} and f_n^{Q4} as n grows large can be obtained by requiring

$$\lim_{n\to\infty} v\big(L_{\mu_1,f_n^{Q3}}, \ldots, L_{\mu_i,f_n^{Q3}}, \ldots\big) = v\big(\lim_{n\to\infty} L_{\mu_1,f_n^{Q3}}, \ldots, \lim_{n\to\infty} L_{\mu_i,f_n^{Q3}}, \ldots\big) \text{ and} \tag{4}$$
$$\lim_{n\to\infty} v\big(L_{\mu_1,f_n^{Q4}}, \ldots, L_{\mu_i,f_n^{Q4}}, \ldots\big) = v\big(\lim_{n\to\infty} L_{\mu_1,f_n^{Q4}}, \ldots, \lim_{n\to\infty} L_{\mu_i,f_n^{Q4}}, \ldots\big).$$

Consequently, an agent with a smooth multi-prior representation (subject to certain con-

[4] Some restriction on $v(\dot{)}$ or on the set of μ_i's usually apply; otherwise the representation in (3) is vacuous in that one can represent any function, $V(f)$, using the set of point-mass priors.

tinuity properties) will exhibit source indifference when assessing bets over almost objective events.[5]

The main contribution of this paper is to establish that, under mild behavioral conditions, a representation via a restricted set of smooth priors is also *necessary* if the DM's behavior is to exhibit indifference to the source of 'almost objective' events. In a second result, necessary and sufficient conditions are stated that uniquely pin down the representing set of smooth priors that simultaneously characterizes the agent's incomplete comparative likelihood rankings of events (see Axiom P4* of Machina and Schmeidler, 1992; and Nehring, 2001 and 2006). Specifically, there is a unique set of priors, Q, corresponding to the representation in Eq. (3), *and* having the property that the DM always prefers a more desirable payoff in event E rather than E' (regardless what is paid outside $E \cup E'$), if and only if $\mu(E) \geq \mu(E')$ for every $\mu \in Q$.

If one accepts the paper's primitive assumptions, then the modeling of Knightian attitudes toward uncertainty reduces to the selection of a set of priors possessing certain properties (i.e., smoothness of the μ_i's), and a function that aggregates their implied payoff distributions (i.e., $v(\cdot)$), subject to certain continuity properties.

Section 2 introduces the formal setting of the model, including definitions of key concepts such as multi-prior representations, generalized almost objective events, almost objective fair bets, and smooth measures. Section 3 presents the key assumptions, including the notion of indifference to the source of 'almost objective' events, and the main model result concerning the existence of a multi-prior representation. This is then followed by the result concerning the uniqueness of the set of priors and some discussion of the results. Section 4 analyzes some common models for decision making under uncertainty that can (or cannot) be made compatible with the notion of indifference to the source of 'almost objective' events. Section 5 concludes.

[5]Here too, the assumed continuity condition in (4) is weaker than Machina's (2004, Theorem 8) requirement that $v(\cdot)$ be event-smooth. The relationship between Machina's result and the more general ones derived in this paper is further explored in Section 4.2.

2 Preliminaries

Let the state space, S, be a convex subset of $\mathbb{R}^D$, where D is an index set of dimensions ranging over the elements of a (possibly uncountable) separable Hausdorff space. This is sufficiently general to include all Euclidean spaces, the sample space generated by vector-valued sequences, sample spaces generated by multi-variate and continuous-time Itô and/or Poisson processes, etc. Although one can contemplate richer state spaces, this restriction of S lends itself naturally to defining a class of 'smooth' measures. Intuitively, each dimension of S corresponds to a distinct source of uncertainty (e.g., the n^{th} digit of the DJIA at the opening of trading tomorrow, at the close of the next trading day, etc.) and one can make 'odd' or 'even' types of bets on any combination of such sources (e.g., the DJIA at the opening of each of the next 15 consecutive trading days).

Let Σ be the Borel σ-algebra on S in the product topology of $\mathbb{R}^D$, and X be a connected and compact metric space of outcomes with the metric denoted by $||\cdot,\cdot||_X$. The space of acts, $\mathcal{F}$ corresponds to all finite-ranged and Σ-measurable mappings from S to X. The decision maker's (DM) preferences correspond to a reflexive, transitive and complete binary relation, $\succeq$, over $\mathcal{F}$. Let $\Delta_0(X)$ be the space of simple (finite) lotteries on X, and let $P(\Sigma)$ denote the space of all finitely additive probability measures on Σ.

For any collection of pairwise disjoint events, $E_1, E_2, ..., E_n \subset S$, and $f_1, f_2, ..., f_n, g \in \mathcal{F}$, let $f_1E_1f_2E_2...f_nE_ng$ denote the act that pays $f_i(s)$ if the true state, $s \in S$, is in E_i, and pays $g(s)$ otherwise. $E \in \Sigma$ is *null* if $fEh \sim gEh\ \forall f, g, h \in \mathcal{F}$. As usual, $x \in X$ denotes both an outcome as well as the constant act that always pays x. A sequence of acts, $\{f_n\}$, converges *pointwise* to $f \in \mathcal{F}$ whenever $||f_n(s), f(s)||_X \to 0$ for every $s \in S$. Refer to the corresponding topology as the *topology of pointwise convergence on* $\mathcal{F}$.

It is useful to have a definition for a 'smooth' measure in a non-Euclidean space where the Lebesgue measure may not be defined.

Definition 1. For any $\{d_1, ..., d_k\} \subseteq D$ with k finite, $F_{d_1,...,d_k} : S \mapsto \mathbb{R}^k$, where $F_{d_1,...,d_k}(x) = (x^{d_1}, ..., x^{d_k})$, is the $\{d_1, ..., d_k\}$-*projection* of S.

Definition 2. $\mu \in P(\Sigma)$ is said to be *smooth* whenever it is countably additive and $\hat{\mu}(B) \equiv \mu\big(F^{-1}_{d_1,...,d_k}(B)\big)$ is absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^k$, where $F_{d_1,...,d_k}$ is a $\{d_1, ..., d_k\}$-projection of S and B is a Borel set in $\mathbb{R}^k$.

Definition 3. $\mu \in P(\Sigma)$ is said to be mutually absolutely continuous with $\succeq$ whenever E is null iff $\mu(E) = 0$.

It is shown in the proof to Proposition 2 that $\mathbb{R}^D$ is separable in its product topology. Thus S too must be separable, and one can constructively show that it admits a smooth measure.[6] An important example is the Wiener measure with a uniformly distributed initial point, defined on the space of vector-valued continuous functions on $[0, 1]$ (see Bass, 1995, for details).

Definition 4. For any $f \in \mathcal{F}$ and $\mu \in P(\Sigma)$, the lottery $L_{\mu,f} \in \Delta_0$ associates the probability $L_{\mu,f}(x) \equiv \mu(f^{-1}(x))$ with outcome x.

The lottery $L_{\mu,f}$ applies μ, a prior distribution on measurable events, to a profile of payoffs (the act f) to produce a distribution over payoffs. Savage's (1954) subjective expected utility theory provides sufficient conditions for a utility representation of the form

$$f \succeq g \;\Leftrightarrow\; \int u \; dL_{\mu,f} \geq \int u \; dL_{\mu,g}.$$

In the above equation, u is a cardinal utility function over outcomes in X and $\int u dL_{\mu,f}$ is the expected utility of the lottery distribution, $L_{\mu,f}$. The multi-prior maximin representation of Gilboa and Schmeidler (1989) takes the form:

$$f \succeq g \;\Leftrightarrow\; \inf_{\mu \in Q} \int u \; dL_{\mu,f} \geq \inf_{\mu \in Q} \int u \; dL_{\mu,g},$$

where $Q \subset P(\Sigma)$. More generally, a multi-prior utility representation takes the form,

$$f \succeq g \;\Leftrightarrow\; v\Big(\Big[L_{\mu,f}\Big]_{\mu \in Q}\Big) \geq v\Big(\Big[L_{\mu,g}\Big]_{\mu \in Q}\Big).$$

[6]The construction follows closely the discussion of non-denumerable measures on product spaces in standard textbooks. See, for example, Ash (1972) and Parthasarathy (1967).

One could think of $\Big[L_{\mu,f}\Big]_{\mu\in Q}$ as a vector with coordinates indexed by $\mu \in Q$, and where each component takes its value in Δ_0. To facilitate notation, refer to this 'vector' as $\Lambda_f^Q \equiv \Big[L_{\mu,f}\Big]_{\mu\in Q}$, and let $\Lambda_f^Q(\mu) \equiv L_{\mu,f}$ correspond to a particular 'coordinate' (i.e., Λ_f^Q is a function from Q to Δ_0). A general multi-prior representation therefore takes the form,

$$f \succeq g \;\Leftrightarrow\; v\Big(\Lambda_f^Q\Big) \geq v\Big(\Lambda_g^Q\Big),$$

for some $Q \subset P(\Sigma)$. It should be emphasized that, in its full generality, a multi-prior representation imposes little restriction. Specifically, every function $V(f)$ can be recast using a multi-prior representation by letting Q be the set of all point-masses. Thus, for a multi-prior representation to be meaningful, either the set Q or the manner in which $v(\cdot)$ aggregates the $L_{\mu,f}$'s must satisfy some restrictions.

Fixing Q, the sequence, $\{\Lambda_{f_n}^Q\}$ is said to converge pointwise to some Λ_f^Q whenever $\Lambda_{f_n}^Q(\mu)$ weakly converges in distribution to $\Lambda_f^Q(\mu)$ for every $\mu \in Q$. The natural domain for v is the set of all Λ_f^Q's that are generated by acts $f \in \mathcal{F}$:

$$\mathcal{L}(Q) = \{\Lambda_f^Q \mid f \in \mathcal{F}\},$$

Thus, a multi-prior representation is simply a function, $v : \mathcal{L}(Q) \mapsto \mathbb{R}$ for some $Q \subset P(\Sigma)$. Moreover, such a representation is said to be continuous if $v(\Lambda_n^Q) \to v(\Lambda^Q)$ whenever the sequence $\{\Lambda_n^Q\} \subset \mathcal{L}(Q)$ converges pointwise to $\Lambda^Q \in \mathcal{L}(Q)$. The representation is said to be monotonic if $v(\Lambda_f^Q) > v(\Lambda_g^Q)$ whenever $x \succ y$ while for every $\mu \in Q$, $\mu(f^{-1}(x)) > \mu(g^{-1}(x))$, $\mu(f^{-1}(y)) < \mu(g^{-1}(y))$, and $\mu(f^{-1}(z)) = \mu(g^{-1}(z))$ for all $z \neq x, y$ in X; i.e., f is preferred to g whenever it is the case that for every $\mu \in Q$, the lottery generated from f coincides with the lottery generated from g, except that mass is shifted from x to y.

Finally, a set of measures, $Q \subset P(\Sigma)$, is said to be *convex ranged* if for every $\alpha \in (0, 1)$ and event, $E \in \Sigma$, there is a subevent, $e \subset E$, with $\mu(e) = \alpha\mu(E)$ for every $\mu \in Q$.

2.1 'Almost objective' binary acts

If one imagines the stripes becoming increasingly finer, Figure 2 illustrates almost objective events constructed along two different directions. In principle, one can construct a similar object along any single dimension indexed by D, or even using multiple dimensions (e.g., a checkerboard pattern). Say that $I_1 \times ... \times I_k$ is a k-cube of size s whenever each of the I_j's is an interval in $\mathbb{R}$ with length s. A k-cube, C, of size s has volume $m_k(C) = s^k$, where m_k is the Lebesgue measure in $\mathbb{R}^k$.

Definition 5. Let $\{B_n\}$ be a sequence of subsets of $\mathbb{R}^k$, $k < \infty$, having the property:

$$\lim_{n\to\infty} s^{-k} m_k\Big(B_n \cap C_s\Big) \to \frac{1}{2}, \tag{5}$$

for any arbitrary k-cube, C_s, of size s. Then for any $\{d_1, ..., d_k\} \subset D$, the sequence $\{E_n = F^{-1}_{d_1,...,d_k}(B_n)\} \subset \Sigma$ is said to approach an *almost objective ethically neutral event.*

The key to viewing E_n and its complement as approaching an ethically neutral event is that in each 'cylinder', $F^{-1}_{d_1,...,d_k}(C_s)$, the 'density' of E_n and its complement approach $\frac{1}{2}$ as n grows large. This aspect is explicitly recognized in Definition 5.[7] For example, consider the hatched region in the left act depicted in Figure 2. Each of the B_n's corresponds to the union of alternating intervals in $\mathbb{R}$ (i.e., along the Nikkei axis) so that $k = 1$ in this case. One projection mapping (i.e., one instance of $F_{d_1,...,d_k}$) is a simple projection from the rectangle onto the Nikkei axis, and $F^{-1}_{d_1,...,d_k}(B_n)$ is a union of alternating stripes. This definition is a generalization of Machina's (2004) in several respects. First, and most obvious, is the generalization to 'large' spaces. Second, rather than thinking of almost objective ethically neutral events as 'stripes', a more appropriate analogy suggested by Definition 5 is a 'salt and pepper' pattern. To provide more intuition, restrict attention to the case where $S = [0, 1]$, so that D, the index set of the dimensions of the state space, is a singleton. Machina (2004) constructs E_n by partitioning $[0, 1]$ into n equal-length intervals, and then taking

[7] Ramsey (1926) defined an ethically neutral event as one whose truth, in and of itself, is inconsequential to the DM and is believed as likely to be true as it is to be false.

the union of every other interval. While this is consistent with Definition 5, the important characteristic of Machina's almost objective events is not the periodicity or the alternating pattern used in the construction, per se. Rather, it is the fact that in any fixed interval the density of the event is essentially one half. Consider, for instance, the following construction: Partition $[0, 1]$ into n equal-length intervals and use a random coin-toss for each interval to determine whether it is in E_n or its complement. For any Borel set E, the Lebesgue measure of $E_n \cap E$ as n grows large will converge to one-half the Lebesgue measure of E. Thus $\{E_n\}$, so constructed, approaches an almost objective ethically neutral event.

The following result extends Machina's (2004) results to smooth priors in the sense of Definition 2.

Proposition 1. *Let* $\{E_n\} \subset \Sigma$ *approach an almost objective ethically neutral event. If* $\mu \in P(\Sigma)$ *is smooth, then* $\lim_{n\to\infty} \mu(E \cap E_n) = \mu(E)/2$ *for any* $E \in \Sigma$.

Machina (2004) obtains essentially the same result in a finite Euclidean space when μ has a continuous density. Proposition 1 confirms Machina's (2004, see page 31) conjecture that this can be weakened to the requirement of absolute continuity of μ with respect to the Lebesgue measure. I.e., μ has to have a density, but the density need not be continuous.

Definition 6. The sequence $\{f_n\} \subset \mathcal{F}$ is said to approach an *almost objective fair bet* whenever for some $x, y \in X$ each f_n can be written as $f_n = xE_ny$ for some sequence $\{E_n\} \subset \Sigma$ approaching an almost objective ethically neutral event.

It is important to emphasize that the limit of a sequence converging to an almost objective fair bet, even if it exists, is not Σ-measurable.[8]

3 Axioms and Main Result

To begin, some standard assumptions are placed on $\succeq$:

[8]Border, Ghirardato, and Segal (2005) prove the existence of a sub-algebra of Σ that delivers 'objective' events in a subjective setting. Because they cannot actually construct such events, it is not clear how such events might be appropriate for use in stating behavioral Axioms (as, for example, the events in Definition 5 are used in this paper).

Axiom 1. $\succeq$ *is transitive, complete, nontrivial (i.e.,* $\exists f, g \in \mathcal{F}$ *with* $f \succ g$*), and for any* $f \in \mathcal{F}$*, the sets* $\{g \succeq f \mid g \in \mathcal{F}\}$ *and* $\{f \succeq g \mid g \in \mathcal{F}\}$ *are closed in the topology of pointwise convergence on* $\mathcal{F}$*.*

The next result states that Axiom 1 is equivalent to the existence of a continuous utility representation. The proof proceeds by establishing that $\mathcal{F}$ is separable and connected in the topology of pointwise convergence. One can subsequently use the results from Debreu (1954) and Eilenberg (1941).

Proposition 2. $\succeq$ *satisfies Axiom 1 if and only if it has a continuous utility representation,* $V : \mathcal{F} \to \mathbb{R}$*, such that* $V(f) \geq V(g) \Leftrightarrow f \succeq g$*.*

Continuity of $\succeq$ is important so as to be able to extend preferences to include the limits of sequences converging to almost objective fair bets. Next, Savage's monotonicity condition, P3, is imposed.

Axiom 2. *For any non-null event,* $E \in \Sigma$*, act* $f \in \mathcal{F}$ *and any* $x, y \in X$*,* $x \succ y \;\Leftrightarrow\; xEf \succ yEf$*.*

While there are various weaker version of monotonicity available in the literature, adopting Axiom 2 works well with the following simple and intuitive indifference to the source of almost objective fair bets:

Axiom 3. *Suppose* $\{E_n\}$ *and* $\{E'_n\}$ *both approach an almost objective ethically neutral event. If* $h \succeq z(E_n \setminus E)w(E \setminus E_n)f \succeq g$ *for all* n*, then for any* $h' \succ h$*, there exists* N *such that* $h' \succ z(E'_n \setminus E)w(E \setminus E'_n)f$ *for all* $n \geq N$*. Likewise, for any* $g' \prec g$*, there exists* N' *such that* $g' \prec z(E'_n \setminus E)w(E \setminus E'_n)f$ *for all* $n \geq N'$*.*

The first major result can now be stated.

Proposition 3. *Axioms 1-3 imply that* $V(\cdot)$ *in Proposition 2 has the following properties:*

i) If $\{f_n\}$ *is an almost objective fair bet, then* $\lim_n V(f_n)$ *exists.*

ii) If $\{f_n\}$ and $\{f'_n\}$ are almost objective fair bets defined with respect to the same outcomes, then for any $E \in \Sigma$ and $f \in \mathcal{F}$, $\lim_n V(f_n E f) = \lim_n V(f'_n E f)$.

Proof: Consider that the fact that X is a compact and connected metric space, and continuity of $V(\cdot)$, together imply the existence of some $\bar{x}$ (resp., $\underline{x}$) that maximizes (resp. minimizes) $V(\cdot)$ over X. Axiom 2 and non-triviality imply that $\bar{x} > \underline{x}$. Axiom 2 further implies that $\bar{x}$ and $\underline{x}$ maximize and minimize, respectively, $V(\cdot)$ over $\mathcal{F}$. Continuity of $V(\cdot)$ and the connectedness of X additionally imply that any act has a certainty equivalent in X. Now, let $\{f_n\}$ be an almost objective fair bet. Because $V(X)$ is compact and connected in $\mathbb{R}$, $\{V(f_n)\}$ contains a subsequence that converges to $V(x')$ for some $x' \in X$. Moreover, the associated subsequence in $\{f_n\}$ is necessarily an almost objective fair bet. In particular, this is true for every accumulation point of $\{V(f_n)\}$. Axiom 3 implies that no such accumulation point can have utility greater or less than $V(x')$. Thus, because $\{V(f_n)\}$ is contained inside a compact interval of $\mathbb{R}$, and all its accumulation points coincide, $\lim_n\{V(f_n)\}$ exists and equals $V(x')$ for some $x' \in X$. A similar argument establishes Part (ii). □

The proposition establishes, in particular, that there can be no source dependence when choosing among almost objective fair bets: The limits of the utilities of the acts depicted in Figure 2 exist and coincide. To reach the main result, establishing the existence of a multi-prior representation with smooth priors, a final condition is imposed.

Axiom 4. *Suppose $x \succ y$. Then if $E \in \Sigma$ is not null, there exists a finite positive integer, n_E, such that any finite collection, $\{E_1, ..., E_n\} \subset \Sigma$ with $xE_iy \succeq xEy$ $\forall i$, covers some part of S at least n/n_E integer times (rounded up).*[9]

Axiom 4 says that a finite collection of events that dominate E (in the sense that the DM prefers to bet a dominating outcome on any collection member rather than on E), cannot be arbitrarily large without covering some part of S, possibly multiple times. The number, n_E, should be viewed as a covering number for S using events that dominate E and are pair-wise

[9]An equivalent statement is that $\{E_1, ..., E_n\}$ contains a sub-collection $\{\hat{E}_1, ..., \hat{E}_k\}$ with $k \geq n/n_E$ and $\bigcap_{i=1}^{k} \hat{E}_i \neq \emptyset$ (adopting the convention that $\bigcap_{i=1}^{1} \hat{E}_i \equiv \hat{E}_1$).

disjoint. I.e., A partition of S whose members dominate E in the sense described can have no more than n_E elements.

The first key result follows.

Theorem 1. *The following are equivalent:*

i) $\succeq$ *satisfies Axioms 1-4*

ii) $\exists Q \subset P(\Sigma)$ *and a continuous and monotonic representation of* $\succeq$, $v : \mathcal{L}(Q) \mapsto \mathbb{R}$, *having the following properties:*

p1: Measures in Q *are smooth and mutually absolutely continuous with* $\succeq$*, while* Q *is convex and relatively closed with respect to the space of measures that are mutually absolutely continuous with* $\succeq$*.*

p2: For any $g \in \mathcal{F}$*,* $E \in \Sigma$*, and* $\{f_n\}$ *approaching an almost objective fair bet,*

$$\lim_{n\to\infty} v(\Lambda^Q_{f_n E g}) \equiv v(\lim_{n\to\infty} \Lambda^Q_{f_n E g}) \tag{6}$$

is well defined.

Proof of the theorem, presented in the Appendix, proceeds as follows. Axiom 4 implies that events can be ranked according to their covering numbers (the n_E's), and that all non-null events have finite covering numbers. A result due to Kelley (1959) establishes an equivalence between this property and the existence of a finitely additive measure, $\breve{\mu}$, that represents this ranking, and in particular, is absolutely continuous with $\succeq$. Axiom 3, asserting indifference to the source of almost objective ethically neutral events, along with monotonicity (i.e., Axiom 2), can be used to show that $\breve{\mu}$ is non-atomic, countably additive, and smooth. Although these three properties of $\breve{\mu}$ appear intuitive, the proofs are non-trivial. Part (ii) of the Theorem is shown to be true in one special case: when Q is associated with all measures that are mutually absolutely continuous with respect to $\breve{\mu}$. In this case, one can recover any $f \in \mathcal{F}$ from Λ^Q_f up to zero $\breve{\mu}$-measure events. Continuity in $\mathcal{L}(Q)$ follows from the fact that zero $\breve{\mu}$-measure events are null, while the remaining properties follow from an argument similar to that used in proving Proposition 3. In particular, Property $p2$ in the Theorem is essentially the condition considered in the Introduction in the form of Eq. (4). What the Theorem establishes is that no representation of the preferences determined by Eq. (1) in the Introduction can be extended via Eq. (6), and thus represented in the manner derived in Theorem 1.

3.1 Uniqueness of Q

The set Q in Theorem 1 need not be unique. This, however, is not an uncommon situation when one deals with multi-prior representations. Marinacci (2002), for instance, provides an example of a maxi-min multi-prior representation that is probabilistically sophisticated (see also Maccheroni, Marinacci, and Rustichini, 2006), and can therefore be reduced to a single unique prior in the context of a non-expected utility representation. In general, the set of priors used in the multi-prior representation of Gilboa and Schmeidler (1989) is only pinned down uniquely if one demands that the representation be in expected-utility form. Thus, given the weak structure furnished by Axioms 1-4, it should not come as a surprise that Theorem 1 does not uniquely pin down a prior set.

The objective of this subsection, is to impose additional structure so as to establish the existence of a continuous multi-prior representation with respect to a *unique* set of smooth priors, Q, exhibiting properties that render it a set of 'beliefs' in a sense to be described shortly.

Axiom 5. *Suppose* $E, E', E'' \in \Sigma$ *and that* $\forall f \in \mathcal{F}$ *and* $\forall x, y \in X$ *such that* $x \succ y$, $x(E \setminus E')y(E' \setminus E)f \succeq y(E \setminus E')x(E' \setminus E)f$ *and* $x(E' \setminus E'')y(E'' \setminus E')f \succeq y(E' \setminus E'')x(E'' \setminus E')f$. *Then* $x(E \setminus E'')y(E'' \setminus E)f \succeq y(E \setminus E'')x(E'' \setminus E)f \;\; \forall f \in \mathcal{F}$ *and* $\forall x \succ y$ *in* X.

Axiom 6. *Let* $f = x_1 E_1 ... x_{n-1} E_{n-1} x_n$ *and* $g = x_1 E'_1 ... x_{n-1} E'_{n-1} x_n$, $E_n \equiv S \setminus \bigcup_{i=1}^{n-1} E_i$, $E'_n \equiv S \setminus \bigcup_{i=1}^{n-1} E'_i$. *If for every* $i = 1, ..., n$, $h \in \mathcal{F}$ *and* $x, y \in X$ *such that* $x \succ y$, $x(E_i \setminus E'_i)y(E'_i \setminus E_i)h \sim x(E'_i \setminus E_i)y(E_i \setminus E'_i)h$, *then* $f \sim g$.

Axiom 7. *Every* $E \in \Sigma$ *contains a subevent,* $A \in \Sigma$, *such that* $xAy(E \setminus A)f \sim yAx(E \setminus A)f$ *for every* $x, y \in X$ *and* $f \in \mathcal{F}$.

To interpret Axiom 5, consider first the case in which E, E', E'' are pairwise disjoint. Then the axiom states that if the DM always prefers to bet a dominating outcome (i.e., x) on E rather than E', and always prefers to bet a dominating outcome on E' rather than E'', then she always prefers to bet a dominating outcome on E rather than E''. The axiom also asserts this when the pairwise intersections of E, E' and E'' are non-empty. Axiom 6 states that whenever the DM is indifferent to betting a dominating outcome on E_i or E'_i, with each of $\{E_i\}_{i=1}^n$ and $\{E'_i\}_{i=1}^n$ a finite partition of S, then the DM will also be indifferent between the act that assigns x_i to event E_i for each i, and the act that assigns x_i to event E'_i for each i. Finally, Axiom 7 essentially asserts that the DM has access to some 'mixing device' allowing her to split any event into two sub-events that are equally likely in the sense of Machina and Schmeidler (1992). This leads to the next result.

Theorem 2. *The following are equivalent:*

i) $\succeq$ *satisfies Axioms 1-7*

ii) There exists a unique and convex-ranged $Q \subset P(\Sigma)$*, that satisfies condition (ii) in Theorem 1 as well as the following condition: for any disjoint* $E, E' \in \Sigma$*,*

$$xEyE'f \succeq yExE'f \ \ \forall f \in \mathcal{F} \text{ and } \forall x \succ y \text{ in } X \ \Leftrightarrow \ \mu(E) \geq \mu(E') \ \forall \mu \in Q. \quad (7)$$

The condition in Eq. (7) suggests that Q can be interpreted as a set of beliefs, in the sense of P4*—the Comparative Likelihood relation of Machina and Schmeidler (1992). Here, Q provides a representation for an incomplete likelihood relation: $E \succeq^{\ell} E' \ \Leftrightarrow \ \mu(E) \geq \mu(E') \ \forall \mu \in Q$. Moreover, for any $f, g \in \mathcal{F}$, if $f^{-1}(x) \sim^{\ell} g^{-1}(x) \ \forall x \in X$, then $\Lambda_f^Q = \Lambda_g^Q$ and therefore $f \sim g$.

There are several other papers that attempt to characterize a unique set of 'beliefs' (as opposed to a multi-prior representation). Sinischalchi (2005), for instance, defines a prior as 'plausible' if preferences admit an expected utility representation using this prior over a restricted set of acts. Nehring's (2001, 2006) results are closest to Theorem 2. In fact, the proof of uniqueness and convex rangedness of Q relies on some of his methods. Nehring's focus, however, is not a multi-prior representation of preferences; in particular, he eschews the direct association of beliefs with observed choice in the manner described by Eq. (7) in favor of a model where beliefs are epistemic rather than inferred.

Before sketching the proof of Theorem 2, it is worth noting that the mixing events in Axiom 7 are rendered objective when combined with the remaining axioms (and in particular, with Axiom 3). Specifically, the event A posited in Axiom 7 has the property that $\mu(A) = \frac{1}{2}\mu(E)\ \forall \mu \in Q$. Assuming Axioms 1-7, it is straight forward to establish that for any almost objective fair bet, $\{f_n\}$, paying $x, y \in X$ and $h \in \mathcal{F}$,

$$\lim_n v(\Lambda^Q_{f_n E h}) = v(\Lambda^Q_{xAy(E\setminus A)h}). \tag{8}$$

The difficulty of proving Theorem 2 largely lies in establishing sufficiency of the axioms. To do so, one constructs the following incomplete likelihood relation: For any $E, E' \in \Sigma$, let

$$E \succeq^\ell E' \;\Leftrightarrow\; x(E\setminus E')y(E'\setminus E)f \succeq y(E\setminus E')x(E'\setminus E)f \;\; \forall f \in \mathcal{F} \text{ and } \forall x \succ y \in X,$$

In other words, for any two disjoint events, E and E', the DM views E as more likely than E' if she always prefers to bet a dominating outcome on E rather than E'. Axiom 5 implies that $\succeq^\ell$ is transitive. Additivity and monotonicity of $\succeq^\ell$, in the sense of de Finetti, follows by definition and from Axioms 1 & 2.[10] To establish the existence of a set of priors representing $\succeq^\ell$, a set of simple functions is constructed, $\mathcal{E} = \{\chi_E - \chi'_E \mid E \succeq^\ell E'\}$, where χ_E is an indicator function. Letting $\breve{\mu}$ correspond to the smooth measure in the discussion following Theorem 1, it is then shown that the set of extreme points of the closed convex hull of $\mathcal{E}$ in $L_\infty(\breve{\mu})$ is, in fact, $\mathcal{E}$. After some manipulation, standard results in analysis can be invoked to deduce the existence of supporting measures, forming the set Q with its desired properties, that can be used to represent $\succeq^\ell$. Axiom 7 implies that Q is convex ranged. Uniqueness of the set Q, having the desired properties, follows from Nehring (2001, 2006).

To prove that Q can be used to construct a multi-prior representation (as in Theorem 1),

[10] A likelihood relation on an algebra of events Σ is additive whenever $E \succeq^\ell E' \;\Leftrightarrow\; E\cup A \succeq^\ell E'\cup A$ for any $A \cap (E \cup E') = \emptyset$. The likelihood relation is monotonic whenever $S \succ^\ell \emptyset$, and for any $A \in \Sigma$, $S \succeq^\ell A \succeq^\ell \emptyset$.

one need only observe that, by Axiom 6, $\Lambda_f^Q = \Lambda_g^Q \Rightarrow f \sim g$, and thus $\succeq$ induces a binary relation on $\mathcal{L}(Q)$. Continuity of this relation in the topology of pointwise convergence follows from Theorem 1. Use of this continuity property assures the existence of a continuous and monotonic representation, $v : \mathcal{L}(Q) \mapsto \mathbb{R}$.

3.2 Discussion

Theorem 1 establishes that, under mild behavioral conditions, a representation in terms of a set of smooth priors is necessary for indifference to the source of almost objective fair bets. The latter requirement can rule out models of Knightian attitudes toward uncertainty, like the model of Eq. (1). In what sense does the decision rule implied by Eq. (1) fail to be representable via smooth priors? The first two terms pose no difficulty because they correspond to an assessment of payoffs using a single prior that is absolutely continuous with respect to the Lebesgue measure. The third term, however, can be written as follows:

$$\int \kappa(L_{\mu_{i_0},f})\rho_N(i_0)di_0$$

where μ_{i_0} is the measure with density $\rho(i_D|i_N)\delta_i(i_N - i_0)$; $\rho(i_D|i_N)$ is a conditional density, $\delta_i(i_N - i_0)$ is the Dirac distribution corresponding to a unit mass atom at $i_N = i_0$, i_0 ranges over all real values, and $\kappa(L)$ is the standard deviation associated with the lottery L. By virtue of the Dirac distribution, each of the priors used to calculate κ is not smooth.

One could also write

$$\lim_{l\to\infty} \int \kappa(L_{\mu_{i_0,l},f})\rho_N(i_0)di_0$$

where $\mu_{i_0,l}$ is a distribution that is absolutely continuous with respect to the Lebesgue measure and converges to one with density $\rho(i_D|i_N)\delta_i(i_N - i_0)$ as $l \to \infty$. While this alternative representation aggregates over smooth measures, it fails to be continuous in the sense required by the Corollary to Theorem 1. Specifically, consider the act depicted on the left in Figure 2, and let n correspond to the size of the partitions depicted (i.e., number of slices);

refer to this act as f_n. To satisfy Eq. (6), i.e., the Corollary to Theorem 1, it must be that

$$\lim_{n\to\infty}\lim_{l\to\infty}\int\kappa(L_{\mu_{i_0,l,f_n}})\rho_N(i_0)di_0=\lim_{l\to\infty}\int\lim_{n\to\infty}\kappa(L_{\mu_{i_0,l,f_n}})\rho_N(i_0)di_0.$$

This, however, is false because for each n, $\lim_{l\to\infty}\int\kappa(L_{\mu_{i_0,l,f_n}})\rho_N(i_0)di_0=0$ while for each l, $\int\lim_{n\to\infty}\kappa(L_{\mu_{i_0,l,f_n}})\rho_N(i_0)di_0=\$50$. In short, there is no way to rewrite the representation in (1) along the lines specified in Theorem 2.

3.3 Relation to Machina's notion of 'event-smoothness'

The Introduction notes that in Theorem 8 of Machina (2004) it is shown that an everywhere event-smooth representation of $\succeq$ satisfies asymptotic indifference to the source of almost objective fair bets. The purpose of this subsection is to explain the connection between the result in Theorem 1 and Machina's (2004) Theorem 8.

To begin, let S be a subset of a Euclidean space, m the Lebesgue measure on S, and suppose $V(f)$ represents $\succeq$. Then $V(\cdot)$ is *event-smooth* at f whenever $V(\hat{f})$ can be approximated as

$$\begin{aligned}V(\hat{f})=V(f)+\sum_{x\in X}\Big[&\Big(\lambda^A(\Delta E_x^+;x,f)-\lambda^A(\Delta E_x^-;x,f)\Big)A_{x,f}\\&-\Big(\lambda^B(\Delta E_x^+;x,f)-\lambda^B(\Delta E_x^-;x,f)\Big)B_{x,f}\Big]+o\Big(\sum_{x\in X}m\big(\Delta E_x^+\big)\Big),\end{aligned}$$

where $\Delta E_x^+\equiv\hat{f}^{-1}(x)\backslash f^{-1}(x)$, $\Delta E_x^-\equiv f^{-1}(x)\backslash\hat{f}^{-1}(x)$, $\lambda^A(\cdot,x,f)$ and $\lambda^B(\cdot,x,f)$ are smooth probability measures (see Definition 2), $A_{x,f}$ and $B_{x,f}$ are bounded and continuous in x, and $o(z)/z$ approaches zero as z approaches zero.[11]

The domain of the representation in Theorem 1, i.e., $\mathcal{L}(Q)$, is a mixture space. Thus one

[11] A closely allied notion of event-wise differentiability is developed in Epstein (1999).

can define v to have a local linear approximation (i.e., Fréchet derivative) at Λ_f^Q whenever

$$v\left(\Lambda_{\hat{f}}^Q\right) = v\left(\Lambda_f^Q\right) + \int_Q \sum_{x \in X} u(\alpha, x; f)\Delta p(\alpha, x; f, \hat{f}) d\rho(\alpha) + o\Big(\sum_{x \in X} |L_{\mu,\hat{f}}(x) - L_{\mu,f}(x)|\Big),$$

where α ranges over elements of Q, ρ is a probability measure over Q, u is continuous and bounded in its first two arguments, and $\Delta p(\alpha, x; f, \hat{f}) \equiv L_{\mu_\alpha,\hat{f}}(x) - L_{\mu_\alpha,f}(x)$.

The following result helps to clarify the relationship between event-smoothness and the representation implied by Theorem 1.

Lemma 1. *v in Theorem 1 has a local linear approximation at Λ_f^Q if and only if it is event-smooth at f.*

In view of Lemma 1, one can interpret Theorem 8 in Machina (2004) as corresponding to the special situation in Theorem 1 where $v(\cdot)$ has a local linear approximation at every act. It should be clear that there are many models that may violate this requirement (e.g., the maximin and maximax representations discussed earlier). In particular, $\succeq$ can be probabilistically sophisticated, satisfy Axioms 1-4, and yet not have a local linear approximation everywhere by virtue of not being Fréchet differentiable everywhere.

3.4 Alternative primitives

Definitions 2 and 5 rely on the Lebesgue measure as a benchmark for defining smooth priors and almost objective ethically neutral events. Through Axiom 3, this results in preferences that assign no importance to events that are zero measure with respect to Lebesgue (see, especially, parts ii(a-b) of Theorem 1). Thus, one can say that the definitions presuppose that the DM's views on null events are commensurate with Lebesgue null events. This, however, seems natural if S is a convex subset of $\mathbb{R}^D$. Alternative choices of S may necessitate a modification of the primitives. For instance, consider the case in which S is the Cantor set on $[0, 1]$. To accommodate this particular situation, one can use the Cantor measure instead of the Lebesgue measure in Definitions 2 and 5. Likewise, the measure μ in Theorem 1 will have to be absolutely continuous with respect to the Cantor measure. The general message is that the primitives of the model must suit the nature of the state space being modeled.

4 Implications of Theorem 1 for various models

4.1 Maximin Utility

A sequence $\{\mu_n\} \subset P(\Sigma)$ is said to weakly converge to $\mu \in P(\Sigma)$ whenever $\int_S g(s)d\mu_n(s) \rightarrow \int_S g(s)d\mu(s)$ for every bounded and continuous, $g : S \mapsto \mathbb{R}$. Consider the function,

$$v(\Lambda_f^Q) = \min_{\mu \in Q} \int_X u(x)dL_{\mu,f}(x), \tag{9}$$

where $dL_{\mu,f}(x)$ corresponds to the distribution over X induced by $L_{\mu,f}$, while $Q \subseteq P(\Sigma)$ is closed and convex in the topology of weak convergence. This is the maximin multi-prior model of Gilboa and Schmeidler (1989).[12] To be sure that this model is admissible under Theorem 1, several requirements must be met:

Proposition 4. *The model in* (9) *satisfies Axioms 1-4 iff $u(x) > u(y)$ for some $x, y \in X$, all elements of Q are mutually absolutely continuous, and Q contains a smooth measure.*

Proof of Proposition 4: First consider the sufficiency of the axioms. Monotonicity with respect to some $x, y \in X$ implies that $u(x) > u(y)$ for some $x, y \in X$. If $\mu_1, \mu_2 \in Q$ are not mutually absolutely continuous, then there is some $E \in \Sigma$ such that $\mu_1(E) = 0 < \mu_2(E)$. It should be clear that $x \succ yEx$, thus E is not null. It should be equally clear that $xEy \sim y$; thus Axiom 2 is violated unless all elements of Q are mutually absolutely continuous. Theorem 1 implies that Q contains a smooth measure. Necessity is trivial. □

Epstein and Marinacci (2006) show that a weak type of monotonicity condition they term the 'Generalized Kreps Axiom' also implies that all measures in Q must be mutually absolutely continuous.

[12] Another related model is the Ghirardato and Marinacci (2002) α-maximin model. By way of restrictions implied by Theorem 1, much of what is said about one model will apply to the other as well.

4.2 Nau and Klibanoff-Marinacci-Mukerji representations

Consider the function

$$v(\Lambda_f^Q) = \int_Q v_\alpha\Big(u_\alpha^{-1}\int u_\alpha\big(f(s)\big)dL_{\mu_\alpha,f}\Big)d\rho(\alpha),$$

where α ranges over elements of Q, and ρ is a probability measure over Q. This is only slightly more general than the function considered by Klibanoff, Marinacci and Mukerji (2005) and Nau (2001)—their v has no α dependence. Related models are also studied in Giraud (2006) and Seo (2006). While, in principle, Q may contain measures that are not commensurate with the conditions in Theorem 1, it should be clear that such conditions can be easily adopted.

4.3 Source-dependent recursive utility in continuous time

Following the generalization of Duffie and Epstein's (1992) stochastic differential utility by Lazrak and Quenez (2003), Schroder and Skiadas (2003) analyze the following backward stochastic differential equation corresponding to source-dependent recursive utility:

$$\frac{dU_t}{U_t} = -g_t + q_t'\sigma_t^U + \kappa_t'|\sigma_t^U| + \frac{1}{2}\sigma_t^{U\prime}Q_t\sigma_t^U + \sigma_t^{U\prime}dB_t, \;\; U_T = u_T, \tag{10}$$

where B_t defines an N-dimensional Brownian diffusion process; g_t, q_t, κ_t and Q_t are adapted to this diffusion process, and σ_t^U is the vector of volatilities of U_t that has to be solved for simultaneously with U_t.[13] If $\kappa_t = 0$ and $Q_t = 0$ then U_t corresponds to a state-dependent continuous-time version of Kreps-Porteus (1978) preferences. The κ_t term, when added to this, corresponds to a generalization of Chen and Epstein's (2002) inter-temporal maximin model. If κ_{1t} is strictly greater than κ_{2t} then the decision maker will be more risk-averse to to bets contingent on the path of B_{1t} then identically distributed bets on the path of B_{2t}. Chen and Epstein (2002), however, demonstrate that U_t could be written as a maximin expected utility functional with respect to a closed and convex set of mutually absolutely

[13] $|\sigma_t^U|$ is the vector $(|\sigma_{1t}^U|, ..., |\sigma_{Nt}^U|)$.

continuous measures. This set can be shown to be smooth in the sense of Definition 2. Thus, the type of source dependence captured by $\kappa_t \neq 0$ is consistent with Theorem 1.

Schroder and Skiadas (2003) argue that the κ_t term captures first-order risk attitudes (as in Segal and Spivak, 1990). They further point out that adding the Q_t term can capture source-dependent second-order risk attitudes. The following example demonstrates that employing a Q_t term to capture source-dependent risk attitudes is not consistent with Axiom 3.

Example 1. Suppose $N = 2$, and that for an act f that pays $f(B_T) \in [0,1]$ at date T, the utility at date T is $u\big(f(B_T)\big)$, while at dates $t < T$ the utility of f is determined by

$$\frac{dU_t}{U_t} = \Big(\frac{Q_1}{2}(\sigma^U_{1t})^2 + \frac{Q_2}{2}(\sigma^U_{2t})^2\Big)dt + \sigma^{U\prime}_{1t} dB_{1t} + \sigma^{U\prime}_{2t} dB_{2t}. \tag{11}$$

Suppose, further, that $u(0) = 1$ and $u(1) = 2$. Now consider the following almost-objective fair bets, $\{f^i_n\}$, for $i = 1, 2$:

$$f^i_n(B_T) = \begin{cases} 1 & \text{if floor}(10^n B_{iT}) \text{ is even} \\ 0 & \text{if floor}(10^n B_{iT}) \text{ is odd} \end{cases}, \tag{12}$$

where floor(x) is the largest integer smaller than or equal to x. Thus, the payoffs of f^i_n depend only on uncertainty generated by a single source, namely B_{it}.

The following result is due to Mark Schroder.

Proposition 5.

$$\lim_n U_t(f^i_n) = \Big\{\frac{1}{2}(1 + 2^{1-Q_i})\Big\}^{\frac{1}{1-Q_i}}.$$

An elegant proof of this, generously provided by Mark Schroder, is in the Appendix. What the proposition establishes is that at any time before T, the DM views the almost objective fair bet $\{f^i_n\}$ as a binary lottery that pays either 1 or 2 with equal probability, that is resolved only at date T, and about which no learning takes place before date T. Moreover, the DM views this lottery as if he had relative risk aversion of Q_i. If $Q_1 \neq Q_2$, then $\lim_n U_t(f_n^1) \neq \lim_n U_t(f_n^2)$, violating Axiom 3. Thus, in contrast with Chen-Epstein preferences, the preferences defined by Eq. (11) cannot be represented using a smooth multi-prior representation.

4.4 Nau, Ergin-Gul, and Chew-Sagi representations

Theorem 1 appears to impose only mild additional constraints on the models of choice under uncertainty considered in Subsections 4.1 and 4.2, and is entirely consistent with the Chen-Epstein version of the model in Subsection 4.3. The reason for this is that the preceding models are *explicit*, or known to have explicit formulations, in their use of a multiple prior framework, making it a simple matter to incorporate the restrictions implied by Theorem 1. The example in Subsection 4.3 demonstrates that this is not true of all models conforming to Eq. (10). This is also not true of a class of models proposed by Nau (2006), Ergin and Gul (2004), and Chew and Sagi (2003).[14]. Consider the state space $S = Z_1 \times Z_2$, where $Z_1 = [0, 1]$ and Z_2 is a 'separate' copy of $[0, 1]$ (e.g., one is the Nikkei index while the other is the DJIA—both are normalized). Recall that an act is a mapping from states to consequences (which I take to be monetary for simplicity). Evaluate the certainty equivalent of an act as follows:

$$ce(f) = v^{-1}\left(\int_0^1 v\left(u^{-1}\Big(\int_0^1 u\big((f(z_1, z_2)\big)dz_1\Big)\right)dz_2\right) \tag{13}$$

This is a model of 'source-dependent' risk aversion. The DM evaluates acts by first calculating the u-certainty equivalent of payoffs contingent on $z_2 \in Z_2$. Call this z_2-contingent certainty equivalent $ce_u(z_2, f)$. A potentially different function, v, is then used to calculate a certainty equivalent from the $ce_u(z_2, f)$'s. For instance, if $f(z_1, z_2) = q(z_2)$ only varies along the Z_2 dimension, then $ce_u(z_2, f) = q(z_2)$, and $ce(f)$ is $v^{-1}\Big(\int_0^1 v(q(z_2))z_2\Big)$; alternatively, if $\hat{f}(z_1, z_2) = q(z_1)$ varies only along the Z_1 dimension, then $ce_u(z_2, \hat{f}) = u^{-1}\Big(\int_0^1 u(q(z_1))z_1\Big)$

[14]Nau's (2006) paper deals with a discrete state-space where the concept of almost objective events may not be pertinent. The discussion here only applies to his model if the number of states is sufficiently large so that behavior can be well approximated using a continuum.

is constant, and thus equal to $ce(\hat{f})$. In particular, if u is more concave than v, then $ce(f) \geq ce(\hat{f})$. While this can induce uncertainty aversion in the case of the bets in Figure 1, it also produces source-dependent preference among almost objective fair bets (e.g., the bets depicted in Figure 2). For instance, if f_n is the act depicted on the left side of Figure 2, then $ce(f_n)$ is $v^{-1}\left(\frac{v(100)+v(0)}{2}\right)$ independent of n; alternatively, if $\hat{f}_n$ corresponds to the act on the right side of Figure 2, then $ce(\hat{f}_n)$ is $u^{-1}\left(\frac{u(100)+u(0)}{2}\right)$, also independent of n. If u is more concave than v, as required by uncertainty aversion with respect to the DJIA in Figure 1, then the DM would necessarily violate Axiom 3.

In other words, the model in (13) exhibits the same property as the one in Eq. (1): Non-trivial attitudes toward Knightian uncertainty necessitate a violation of Axiom 3. It is important to emphasize, however, that the violation of Axiom 3 does not arise due to the source preference per se. Rather, it comes about as a result of how the source preference is modeled.

5 Conclusions

Machina (2004, 2005) introduced the notion of an 'almost objective' event in a continuous state space—high frequency events in a subjective setting such as 'the realization of the n^{th} decimal place of a stock index.' Payoffs on such events intuitively appear as objective lotteries in the sense that decision makers should not prefer to place bets on any particular digit when n is large *even if the state space is fully subjective.* This paper investigates the implications of the requirement that decision makers are indifferent to the source of 'almost objective' acts. Given other weak assumptions on preferences, the indifference requirement is tantamount to the existence of a multi-prior representation where each prior is absolutely continuous with respect to a smooth measure that represents the null sets of the DM. The utility of a Savage act can be computed by an aggregator of the lotteries produced by each prior distribution over the act.

The main lesson from analyzing various models appearing in the literature can be summarized as follows: Models that are explicit about their use of multiple priors can generally exhibit source preference while made compatible with the requirements of Theorem 1. Models that do not make explicit use of a multi-prior representation may generically fail to satisfy the Axioms in Theorem 1. Hence, modelers of Knightian attitudes (or source preference) toward uncertainty who also wish to accommodate indifference to the source of almost objective fair bets might consider the representation in Theorem 1 as a starting point. In other words, one ought to begin with a set of smooth priors, Q, and then select a 'lottery aggregation' function v, consistent with Theorem 1.

A Proofs

Proof of Proposition 1: Let $\Sigma_0 \equiv \{F^{-1}_{d_1,\ldots,d_k}(B) \mid B \in \mathbb{R}^k \text{ is Borel }, k < \infty\}$. Then Σ_0 is an algebra, and Σ is the smallest sigma-algebra containing Σ_0. Because μ is countably additive, one can find some $\hat{E} \in \Sigma_0$ such that $\mu(E)$ is arbitrarily close to $\mu(\hat{E})$ (see Ash, Theorem 1.3.11). Thus it is sufficient to consider $E \in \Sigma_0$. Moreover, because the set of finite cubes in $\mathbb{R}^k$ is a base for Borel sets in $\mathbb{R}^k$ and μ is countably additive, it is sufficient to consider $E = F^{-1}_{d_1,\ldots,d_k}(C_s)$ where $C_s \subset \mathbb{R}^k$ is an arbitrary k-cube of finite size.

From Definition 5, any almost objective ethically neutral event is the 'limit' of sets of the form $F^{-1}_{d'_1,\ldots,d'_{k'}}(B_n)$ where the B_n's are subsets of $\mathbb{R}^{k'}$, for some finite k'. There is no loss of generality in assuming that the B_n's and C_s can be embedded in the same Euclidean space, so henceforth I shall assume that both D_n and C_s are subsets of $\mathbb{R}^{k''}$, for some $k'' \leq k + k'$. In what follows, let $\hat{C}_s \equiv F_{d_1,\ldots,d_k,d'_1,\ldots,d'_{k'}}\left(F^{-1}_{d_1,\ldots,d_k}(C_s)\right) \subset \mathbb{R}^{k''}$, and note that $\hat{C}_s$ need not be bounded.

By the Radon-Nykodim Theorem and smoothness of μ, $\mu(E) = \int_{\hat{C}_s} F dm_{k''}$, for some Lebesgue measurable positive scalar function, F, and where $m_{k''}$ is the Lebesgue measure in $\mathbb{R}^{k''}$. The result follows immediately from Definition 5 if F is a simple function. While any measurable F can be approximated as the limit of simple functions, it is not immediately clear that the limit $n \to \infty$ in Eq. (5) commutes with the limit of the simple functions. To establish this, use Proposition 4.13 in Royden (1968) to pick a sequence of decreasing numbers, $(\epsilon_i, \delta_i) \to 0$ so that for any measurable $e \subset \mathbb{R}^{k''}$, $m_{k''}(e) < \delta_i$ implies $\int_e F dm_{k''} < \epsilon_i$. For each i, one can apply a k''-dimensional version of Proposition 3.22 from Royden to write $F = F - G_i + G_i$, where $|F - G_i| < \delta_i$ except on a set of measure less than δ_i (where, without loss of generality, take $G_i = 0$), and G_i is simple (and in the case being considered, non-negative). For each i and n, use Proposition 4.7 in Royden to write

$$\int_{B_n \cap C} (G_i - |F - G_i|) dm_{k''} \leq \int_{B_n \cap C} F dm_{k''} \leq \int_{B_n \cap C} (G_i + |F - G_i|) dm_{k''},$$

where C is any finite union of compact k''-cubes in $\hat{C}_s$. This implies,

$$\int_{B_n \cap C} G_i dm_{k''} - \int_C |F - G_i| dm_{k''} \leq \int_{B_n \cap C} F dm_{k''} \leq \int_{B_n \cap C} G_i dm_{k''} + \int_C |F - G_i| dm_{k''},$$

or

$$\int_{B_n \cap C} G_i dm_{k''} - \delta_i m_{k''}(C) - \epsilon_i \leq \int_{B_n \cap C} F dm_{k''} \leq \int_{B_n \cap C} G_i dm_{k''} + \delta_i m_{k''}(C) + \epsilon_i.$$

Let $\hat{E} \equiv F^{-1}_{d_1,\ldots,d_k,d'_1,\ldots,d'_{k'}}(C)$, i.e., the inverse projection of $C \subset \mathbb{R}^{k''}$ into S. Taking the limit as $n \to \infty$,

$$\frac{\mu_i(\hat{E})}{2} - \delta_i m_{k''}(C) - \epsilon_i < \lim_{n \to \infty} \mu(\hat{E} \cap E_n) < \frac{\mu_i(\hat{E})}{2} + \delta_i m_{k''}(C) + \epsilon_i,$$

where $\mu_i(\hat{E}) \equiv \int_C G_i dm_{k''}$. Taking the limit $i \to \infty$ gives $\lim_{n\to\infty} \mu(\hat{E} \cap E_n) = \frac{\mu(\hat{E})}{2}$. Because this is true for any C, the fact that μ is a probability measure and can thus be approximated arbitrarily well using a measure with finite support, implies that $\lim_{n\to\infty} \mu(E \cap E_n) = \frac{\mu(E)}{2}$. □

Proof of Proposition 2: The idea is to show that $\mathcal{F}$ is separable and connected in the topology of pointwise convergence, and then use the results from Debreu (1954) and Eilenberg (1941).

To prove separability, we need to find a dense countable subset, $F \subseteq \mathcal{F}$. The construction is similar to a proof-sketch of the Hewitt-Marczewski-Pondiczery Theorem by Henno Brandsma.[15] Let Z and Y be separable Hausdorff spaces. It is first proved that Y^Z is a separable Hausdorff space. By assumption, Z and Y contain countable dense subsets, Q_Z and Q_Y, respectively. Moreover, because Z is Hausdorff, there is a countable set of open subsets $\mathcal{O}$ of Z such that if $\{z_1, .., z_k\} \subset Q_Z$ is finite, then there is some $E \in \mathcal{O}$ that separates any $z \in Q_Z \setminus \{z_1, .., z_k\}$ from $\{z_1, .., z_k\}$. Fix some $y_0 \in Y$ and define $F \subseteq Y^Z$ as follows: $f \in F$ whenever there is a finite collection, $\{(E_i, q_i)\} \subseteq \mathcal{O} \times Q_Y$, such that the E_i's are pairwise disjoint, $f(z) = q_i$ if $\omega \in E_i$, and $f(z) = y_0$ if $\omega \notin \bigcup_i E_i$. The set F is countable and, as is now shown, dense in Y^Z. To prove this it is sufficient to show that every open set in a basis for Y^Z contains an element of F. A basis for Y^Z in the topology of pointwise convergence is given by sets of the form $U \equiv \prod_{z \in Z} U_z$ with $U_z = Y$ for all but a finite set $I \subseteq Z$, and U_z an open subset of Y otherwise.[16] Fix such a U. Because I is finite and Z is a separable Hausdorff space, there is an $f \in \mathcal{F}$ defined by $\{(E_z, q_z)\}_{z \in I} \subseteq \mathcal{O} \times Q_Y$ where the E_z's are pairwise disjoint, $z \in E_z$, and $q_z \in U_z$ for each $z \in I$. Every arbitrary basis element, U, therefore contains an element of F, meaning that Y^Z is separable. To show that Y^Z is Hausdorff, consider any $\omega, \omega' \in Y^Z$ such that $\omega \neq \omega'$. Then there is some coordinate z such that $\omega(z) \neq \omega'(z)$. Because Y is separable, there is an element of the basis, U, defined above that contains ω and not ω'.

The previous argument establishes that $\mathbb{R}^D$ is a separable Hausdorff space in its product topology, and thus so is S. Identifying Z with $\mathbb{R}^D$ and Y with X, then implies that X^S is a separable Hausdorff space. Moreover, the dense set F constructed above is composed of simple functions, and is thus dense in $\mathcal{F}$. Consequently, $\mathcal{F}$ is separable. The fact that $\mathcal{F}$ is connected follows directly from the assumption that X is connected. □

Proof of Theorem 1: Assume Axioms 1-4. The proof proceeds in several steps.

Step (i): There exists a finitely additive and non-atomic measure, $\check{\mu}$, such that

[15]See http://at.yorku.ca/cgi-bin/bbqa?forum=homework_help_2002;task=show_msg;msg=0474.0002
[16]See, for example, Gamelin and Greene (1999), p.100.

$\check{\mu}(E) = 0$ **iff** $E \in \Sigma$ **is null.**

Let $[E] = \{E' \mid (E \setminus E') \cup (E' \setminus E) \text{ is null }\}$. Then the set $\mathcal{B} = \{[E] | E \in \Sigma\}$ is a Boolean algebra. Moreover, $\mathcal{B} \setminus \{[\emptyset]\} = \bigcup \mathcal{B}_n$ where $\mathcal{B}_n = \{[E] \mid n_E \leq n\}$. Fixing any $x \succ y$ in X, Axiom 4 in conjunction with Theorem 4 from Kelley (1959) implies the existence of a finitely additive measure, $\check{\mu}$, such that $\check{\mu}([E]) = 0$ iff $[E] = [\emptyset]$ (i.e., iff E is null).

Now, suppose $\check{\mu}$ contains an atom, a. Let $\{E_n\}$ be a sequence approaching an almost objective ethically neutral event. Then, for each n, either $\check{\mu}(a \cap E_n) = \check{\mu}(a)$ or $\check{\mu}(a \cap E_n) = 0$. Let E_n^c be the complement of E_n, and define $E_n' = E_n$ if $\check{\mu}(a \cap E_n) = \check{\mu}(a)$ and $E_n' = E_n^c$ otherwise. The sequence $\{E_n'\}$ also approaches an almost objective ethically neutral event. Non-triviality and monotonicity imply $\exists x, y \in X$ with $x \succ y$, so define $f_n = xE_n'y$ and $f_n' = yE_n'x$. Identify E in Axiom 3 with a, h with xay, and g with xay. Then, because $f_n'ay = y \prec xay$ for all n, Axiom 3 is violated. Thus $\check{\mu}$ is non-atomic.

Step (ii): $\check{\mu}$ **in Step (i) is countably additive:** It is sufficient to show that for any sequence $\{E_n\} \subset \Sigma$ such that $E_{n+1} \subseteq E_n$ and $\cap_n E_n = \emptyset$, $\check{\mu}(E_n) \to 0$ (see, for example, Theorem 1.2.8 from Ash, 1972). If $\check{\mu}(E_n) \not\to 0$, then there is some subsequence of decreasing events, $\{E_n'\} \subseteq \{E_n\}$ such that $\check{\mu}(E_n') \geq \epsilon$, for some $\epsilon > 0$ and all n. By Proposition 1 in Kelley (1959) there must be a finite smallest $\bar{n}$ such that $E_k' \in \mathcal{B}_{\bar{n}}$ for all $E_k' \in \{E_n'\}$. Moreover, because $\check{\mu}$ has no atoms, one can always find some non-null $A \in \Sigma$ such that $A \notin \mathcal{B}_{\bar{n}}$. In particular, $xE_k'y \succeq xAy$ for all $E_k' \in \{E_n'\}$ (otherwise $E_k' \notin \mathcal{B}_{\bar{n}}$ for some k). Thus $V(xE_k'y) \geq V(xAy)$ for all $E_k' \in \{E_n'\}$. On the other hand, $\bigcap_n e_n = \emptyset$, implying that for every $s \in S$, there must be some k such that $s \notin E_k'$. In turn, it must be that $xE_k'y$ converges pointwise to y, thus by Proposition 2, $V(xE_k'y) \to V(y)$—contradicting $xAy \succ y$.

Step (iii): $\check{\mu}$ **in Steps (i) and (ii) is smooth:** Suppose $\check{\mu}$ is not smooth. Then $\check{\mu}_k(B) \equiv \check{\mu}\big(F^{-1}_{d_1,\dots,d_k}(B)\big)$, where B is a Borel set in $\mathbb{R}^k$ and $d_1, \dots, d_k \in D$, is not absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^k$. One can therefore write $\check{\mu}_k = \eta_A + \eta_S$, where η_A is absolutely continuous with respect to the Lebesgue measure, while η_S is singular with respect to Lebesgue. Because both η_A and η_S are countably additive, there must be some finite-sized cube, $C \subseteq F_{d_1,\ldots,d_k}(S)$, such that η_S is singular on C, meaning that there is a subset $c \subset C$, such that $m_k(c) = 0$ while $\eta_S(c) > 0$. Now construct a sequence that approaches an almost objective ethically neutral event as follows: at the n^{th} stage, partition $\mathbb{R}^k$ into slices of the form $I \times \mathbb{R}^{k-1}$, where I is an interval of size 2^{-n}. Let $\hat{E}_n$ consist of the union of the slice containing the origin, and every alternate slice. Define $E_n = F^{-1}_{d_1,\ldots,d_k}\Big(\hat{E}_n \cap F_{d_1,\ldots,d_k}(S) \setminus c\Big)$. It should be clear that $\{E_n\}$ approaches an almost objective ethically neutral event. Let $f_n = xE_ny$ and $f'_n = yE_nx$, where $x \succ y$ and $x, y \in X$, and set $A = F^{-1}_{d_1,\ldots,d_k}(c)$. Then similarly to the proof of non-atomicity in Step (i), $f_nAy = xAy \succ f'_nAy = y$, contradicting Axiom 3.

Step (iv). $\succeq$ can be represented by some continuous and monotonic function $v : \mathcal{L}(Q) \mapsto \mathbb{R}$, where Q consists of measures mutually absolutely continuous with $\breve{\mu}$.

It is first demonstrated that a monotonic multi-prior representation exists that uses all measures that are absolutely continuous with respect to $\breve{\mu}$. The latter set is denoted $\hat{Q}$ and is homeomorphic to $L_1(\breve{\mu})$. Fix $f \in \mathcal{F}$ and denote $p(\mu, x) \equiv \mu\big(f^{-1}(x)\big)$ with $\mu \in \hat{Q}$. Consider a collection of events, $\{E_\mu\}_{\mu\in\hat{Q}}$, and for each μ a single event, E_μ, is chosen so that $\mu(E_\mu) = p(\mu, x)$. Clearly, one can create many such collections. Denote the set of all such collections as $\mathcal{C}(\Lambda^{\hat{Q}}_f, x)$ and consider that any

$$E \in \underset{\{E_\mu\}_{\mu\in\hat{Q}} \in \mathcal{C}(\Lambda^{\hat{Q}}_f, x)}{\operatorname{argmax}} \breve{\mu}\Big(\bigcap_\mu E_\mu\Big)$$

is, up to a null set of $\breve{\mu}$, equal to $f^{-1}(x)$. Thus, given only the set of lotteries, $\Lambda^{\hat{Q}}_f$, one can reconstruct (up to null sets) the act f. Define Q as the set of all measures that are mutually absolutely continuous with $\breve{\mu}$. Because every lottery in $\Lambda^{\hat{Q}}_f$ is arbitrarily close to a lottery in Λ^Q_f, the latter can also be used to construct f up to null sets. There is therefore no ambiguity in setting $v(\Lambda^Q_f) = V(f)$, where $V(\cdot)$ is from Proposition 2.

To show that v is continuous, consider a sequence $\{L_{\mu,g_n}\}$ that weakly converges in distribution to $L_{\mu,g}$, where $\{g_n\}$ and g are in $\mathcal{F}$. If $V(g_n)$ does not converge to $V(g)$, then there is a subsequence of $\{g_n\}$, say $\{g'_n\}$, such that $|V(g'_n) - V(g)| > \eta > 0$ for all n. A minor extension of Theorem 2.5.3 in Ash (1972) (its proof applies more generally to separable metric spaces) implies that $\{g'_n\}$ contains another subsequence, say $\{\hat{g}_n\}$, that converges $\breve{\mu}$-a.e. to g. It must be that $\{\hat{g}_n\}$ converges to $V(g)$, because points of non-convergence form a null set of $\breve{\mu}$ and therefore of $\succeq$, and this contradicts $|V(g'_n) - V(g)| > \eta > 0$. Thus $V(g_n) \to V(g)$, and $v(\Lambda^Q_{g_n}) \to v(\Lambda^Q_g)$ follows by definition. Monotonicity is an immediate consequence of Axiom 2, while Property *p2* follows directly from Proposition 3 and the fact that for any E, Q contains a member that assigns a measure to E that is arbitrarily close to one.

Necessity of Axioms 1-3 is trivial. Axiom 4 follows from Theorem 4 in Kelley (1959). $\square$

Proof of Theorem 2: Sufficiency of the axioms is proved in two steps.

Step (i): There exists a unique and convex-ranged set of priors, Q, that is convex, relatively closed in the set of mutually absolutely continuous measures with $\succeq$, and satisfies Eq. (7)**.**

Begin by defining an incomplete likelihood relation on Σ. For any $E, E' \in \Sigma$, let

$$E \succeq^{\ell} E' \;\Leftrightarrow\; x(E \setminus E')y(E' \setminus E)f \succeq y(E \setminus E')x(E' \setminus E)f \;\forall f \in \mathcal{F} \text{ and } \forall x \succ y \text{ in } X,$$

Axiom 5 implies that $\succeq^{\ell}$ is transitive, additivity of $\succeq^{\ell}$ follows by definition, while monotonicity is a consequence of Axioms 1 and 2 (see Footnote 10). Thus $\succeq^{\ell}$ is an incomplete likelihood relation (i.e., it satisfies the de Finetti definition of a likelihood relation except for completeness).

Next, define

$$\mathcal{E} \equiv \left\{ \chi_E - \chi_{E'} \mid E \succeq^{\ell} E' \right\}.$$

Additivity insures that this set is well defined, while $E \succ^{\ell} \emptyset$ implies that $\chi_E \in \mathcal{E}$ for every non null $E \in \Sigma$. Because $q, q' \in \mathcal{E}$ should be identified whenever they differ by a null set of $\succeq$, one can take $\mathcal{E}$ to be a non-empty subset of $L_\infty(\check{\mu})$, where $\check{\mu}$ is the measure from the proof of Theorem 1. Note that under this identification, the zero function is in $\mathcal{E}$, and $\mathcal{E}$ is closed in the weak* topology of $L_\infty(\check{\mu})$. Moreover, because each $q \in \mathcal{E}$ is contained in the closed unit ball of $L_\infty(\check{\mu})$ (in the sup norm), Alaoglu's Theorem implies that $\mathcal{E}$ is weak* compact. Denote the convex hull of $\mathcal{E}$ as conv($\mathcal{E}$). Because each $q \in \mathcal{E}$ is an extremal point of conv($\mathcal{E}$), the Krein-Milman Theorem implies that conv($\mathcal{E}$) is closed and $\mathcal{E}$ is the set of its extremal points. Consider now the linear cone generated by conv($\mathcal{E}$), and denote it by $\mathcal{C}(\mathcal{E})$. It should be clear that $\mathcal{C}(\mathcal{E})$ is convex, sequentially closed, and that ε is an extremal point of $\mathcal{C}(\mathcal{E})$ if and only if $\varepsilon = \xi q$, with $\xi \in \mathbb{R}_+$ and $q \in \mathcal{E}$. It is next shown that $\mathcal{C}(\mathcal{E})$ is also weak* closed. To this end, note that by Alaoglu's Theorem, $B_r(L_\infty(\check{\mu}))$ is compact and therefore metrizable (and therefore sequentially closed) in the weak* topology of $L_\infty(\check{\mu})$; thus, because $\mathcal{C}(\mathcal{E})$ is sequentially closed, for any $r > 0$, $\mathcal{C}(\mathcal{E}) \cap B_r(L_\infty(\check{\mu}))$ is weak* sequentially closed. Metrizability of $B_r(L_\infty(\check{\mu}))$ further implies $\mathcal{C}(\mathcal{E}) \cap B_r(L_\infty(\check{\mu}))$ is also weak* closed. Given that r is arbitrary, the Krein-Smulian Theorem implies that $\mathcal{C}(\mathcal{E})$ is weak* closed.[17]

Theorem 2 and its corollaries from Phelps (1964) guarantees that $\mathcal{C}(\mathcal{E})$ is the intersection of half-spaces $\{\mathcal{E}_\alpha\}$ with $q \in \mathcal{E}_\alpha \Leftrightarrow \int_S \psi_\alpha q d\check{\mu} \geq \beta_\alpha$, for some non-zero $\psi_\alpha \in L_1(\check{\mu})$ and $\beta_\alpha \in \mathbb{R}$. Because $\mathcal{C}(\mathcal{E})$ is a cone, $\beta_\alpha = 0$. Moreover, because $\chi_E \in \mathcal{C}(\mathcal{E})$ for every $E \in \Sigma$ (up to null sets of $\check{\mu}$), it must that $\psi_\alpha \geq 0$ on S, while $\psi_\alpha > 0$ on some positive $\check{\mu}$-measure subset of S. Under these considerations, one can, without loss of generality, normalize ψ_α so that $\int_S \psi_\alpha d\check{\mu} = 1$. Recalling that ε is an extremal point of $\mathcal{C}(\mathcal{E})$ if and only if $\varepsilon = \xi q$, with $\xi \in \mathbb{R}_+$ and $q \in \mathcal{E}$, and denoting the set of supporting ψ's as $\hat{\Psi}$, one arrives at

$$E \succeq^\ell E' \Leftrightarrow \int_E \psi d\check{\mu} \geq \int_{E'} \psi d\check{\mu} \; \forall \psi \in \hat{\Psi}. \tag{14}$$

The above representation can be restated with $\Psi \equiv \overline{\text{co}}(\hat{\Psi})$, where $\overline{\text{co}}$ denotes a closed convex hull in $L_1(\check{\mu})$.

Define $\hat{Q} \equiv \{\mu \mid \mu(E) = \int_E \psi d\check{\mu},\ \psi \in \Psi\}$. Axiom 4 implies that for every $E \in \Sigma$ one can

[17] The Krein-Smulian Theorem states that because $\mathcal{C}(\mathcal{E})$ is convex and $L_\infty(\check{\mu})$ is the dual of a separable Banach space, $\mathcal{C}(\mathcal{E})$ is weak* closed if and only if $\mathcal{C}(\mathcal{E}) \cap B_r(L_\infty(\check{\mu}))$ is weak* closed for every $r > 0$, where $B_r(L_\infty(\check{\mu}))$ is a closed (in the norm topology) ball of radius r.

find disjoint events $E_1, E_2 \subset E$ such that $\mu(E_1) = \mu(E_2)\ \forall \mu \in Q$. Since each of the E_i's can be split in a similar manner, Axiom 3 (i.e., transitivity of $\succeq^{\ell}$) implies that Q is dyadically convex-ranged in the sense of Nehring (2001, 2006). Theorem 2 in the latter ensures that $\hat{Q}$ is unique (up to the $\overline{\text{co}}$ operator) .

Now, $\hat{Q}$ contains a member that is mutually absolutely continuous with $\succeq$ (or equivalently, with $\breve{\mu}$). To see this, define $E_{\psi}^{\perp} \equiv S \setminus \text{supp}(\psi)$, where $\text{supp}(\psi)$ is the support of $\psi \in \Psi$. Let $\epsilon = \min_{\psi \in \Psi} \breve{\mu}(E_{\psi}^{\perp})$ and suppose that $\hat{Q}$ does not contain a measure that is mutually absolutely continuous with $\breve{\mu}$ (i.e., $\epsilon > 0$). Then there is some ψ^* such that $\breve{\mu}(E_{\psi^*}^{\perp}) = \epsilon$. Because $E_{\psi^*}^{\perp}$ is not null, there is some $\psi \in \Psi$ for which $\int_{E_{\psi^*}^{\perp}} \psi d\breve{\mu} > 0$, and thus $\breve{\mu}(E_{\frac{1}{2}\psi^* + \frac{1}{2}\psi}^{\perp}) < \epsilon$, a contradiction. Let $\mu' \in \hat{Q}$ be mutually absolutely continuous with $\succeq$. Then because $\hat{Q}$ is convex, if $\mu \in \hat{Q}$ is not mutually absolutely continuous with $\succeq$, one can find another measure, say $\alpha\mu + (1-\alpha)\mu' \in \hat{Q}$ that is arbitrarily close to μ (in the $L_1(\breve{\mu})$ sense) and yet mutually absolutely continuous with $\succeq$. One can therefore represent $\succeq^{\ell}$ using only measures that are mutually absolutely continuous with $\succeq$. Setting $Q \equiv \{\mu \in \hat{Q} \mid \mu \text{ is mutually absolutely continuous with } \succeq\}$ completes the proof of this step.

Step (ii): $\succeq$ can be represented by some continuous and xy-monotonic function $v : \mathcal{L}(Q) \mapsto \mathbb{R}$.

Define $\mathcal{L}(Q)$ as in the text. Now, suppose that $L_{\mu,f_n} \to L_{\mu,f}$ for every $\mu \in Q$. Then an argument similar to that given in part (iv) of the proof to Theorem 1 implies that $f_n \to f$ on all but a null set with respect to *all* $\mu \in Q$. Because the representation in (14), Theorem 1 and Axiom 4 imply that E is null iff E has zero measure with respect to all $\mu \in Q$, it must be that $L_{\mu,f_n} \to L_{\mu,f}$ for every $\mu \in Q$ iff $L_{\breve{\mu},f_n} \to L_{\breve{\mu},f}$. Thus, $\succeq$ is continuous in the pointwise topology of $\mathcal{L}(Q)$.

From step (i) and Axiom 6, it must be that for any $f, g \in \mathcal{F}$, $\Lambda_f^Q = \Lambda_g^Q \Rightarrow f \sim g$. Thus $\succeq$ is a an ordering on $\mathcal{L}(Q)$ and, by the previous paragraph, is continuous in its pointwise topology. The remaining properties of v described in Theorem 1 follow from the same arguments made in the proof of Theorem 1.

Necessity of the axioms is straight forward. □

Proof of Proposition 5: Because $\{f_n^i\}$ depends only on the Brownian filtration generated by dB_{it}, the backward stochastic differential equation corresponding to $U_t(f_n^i)$ is given by:

$$\frac{dU_t}{U_t} = \frac{Q_i}{2}(\sigma_{it}^U)^2 dt + \sigma_{it}^U dB_{it}.$$

By Itô's Lemma, $U_t^{1-Q_i}$ is a local martingale, and

$$U_t = \begin{cases} \left(E_t\left[u\left(f_n^i(B_T)\right)^{1-Q_i}\right]\right)^{\frac{1}{1-Q_i}} & \text{if } 0 \le Q_i \ne 1 \\ \exp\left(E_t\left[\ln u\left(f_n^i(B_T)\right)\right]\right) & \text{if } Q_i = 1 \end{cases}. \tag{15}$$

Only the $Q \neq 1$ is explicitly considered. Now, consider that $u\big(f_n^i(B_T)\big) = u\big(f_n^i(B_T - B_t)\big)$ if $10^n B_t$ is an even integer, and $u\big(f_n^i(B_T)\big) = u\big(f_n^i(B_t - B_T)\big)$ if $10^n B_t$ is an odd integer. Because, conditional on time-t information, $B_T - B_t \overset{d}{=} B_t - B_T$, under the assumption that $10^n B_t$ is an integer, one can therefore write

$$\begin{aligned} E_t\left[u\left(f_n^i(B_T)\right)^{1-Q_i}\right] &= \frac{1}{2}\left(E_t\left[u\left(f_n^i(B_T - B_t)\right)^{1-Q_i}\right] + E_t\left[u\left(f_n^i(B_t - B_T)\right)^{1-Q_i}\right]\right) \\ &= \frac{1}{2}E_t\left[u\left(f_n^i(B_T - B_t)\right)^{1-Q_i} + u\left(f_n^i(B_t - B_T)\right)^{1-Q_i}\right] \\ &= \frac{1}{2}\left(1 + 2^{1-Q_i}\right), \end{aligned} \tag{16}$$

where the last equality arises from the fact that floor(x) has the opposite parity of floor$(-x)$. Now, in case that $10^n B_t$ is not an integer, then defining the stopping time, $\tau_n = \inf\{s \geq t \mid 10^n B_s \text{ hits an integer}\}$ and $P(\tau_n < T)$ to be the probability that an integer is hit before T, one easily calculates that

$$E_t\left[u\left(f_n^i(B_T)\right)^{1-Q_i}\right] = \begin{cases} 2^{1-Q_i}\big(1 - P(\tau_n < T)\big) + \frac{1}{2}P(\tau_n < T)(1 + 2^{1-Q_i}) & \text{floor}(10^n B_t) \text{ is even} \\ \big(1 - P(\tau_n < T)\big) + \frac{1}{2}P(\tau_n < T)(1 + 2^{1-Q_i}) & \text{floor}(10^n B_t) \text{ is odd.} \end{cases}$$

Given that $\lim_n P(\tau_n < T) = 1$, one obtains the needed result,

$$\lim_n E_t\left[u\left(f_n^i(B_T)\right)^{1-Q_i}\right] = \frac{1}{2}(1 + 2^{1-Q_i})$$

from which the proposition follows. □

Proof of Lemma 1: Let $A_{x,f} \equiv \int_Q \sum_{x\in X} u(\alpha, x; f)\mathbf{1}_{u(\alpha,x;f)}d\rho(\alpha)$, where 1_z is an indicator function, and $B_{x,f} \equiv \int_Q \sum_{x\in X} u(\alpha, x; f)\mathbf{1}_{-u(\alpha,x;f)}d\rho(\alpha)$. Setting

$$\lambda^A(E; x, f) \equiv \frac{1}{A_{x,f}} \int_Q \sum_{x\in X} u(\alpha, x; f)\mathbf{1}_{u(\alpha,x;f)}L_{\mu_\alpha,xEf}(x)d\rho(\alpha),$$

and similar for $\lambda^B(E; x, f)$ consistently identifies the first-order terms in the two approximations. Part ii(a) of Theorem 1 ensures that the second order terms are comparable as $\sum_{x\in X} m\left(\Delta E_x^+\right)$ approaches zero. □

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