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Convertible debt: Valuation and conversion in complex capital structures

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The impact of conversion strategies on the value of convertible debt is examined. It is shown that, for capital structures comprised of multiple classes of convertible debt, the optimal conversion policy for a given bond class is affected by the decisions of other bond classes. Using this optimal conversion strategy, closed-form solutions for bond prices are derived.

1. Introduction

Many publicly traded firms have complex capital structures comprised not only of debt and equity but also various contingent claims.[1,2] As a consequence, the valuation of corporate securities has received much attention.[3] While current research provides interesting valuation insights, it understates the importance of the interdependence among securities with embedded options. As the warrant valuation literature has shown [Emanuel (1983), Constantinides (1984) and Spatt and Sterbenz (1988)], a strategic dimension is introduced which significantly complicates valuation when corporate securities contain embedded options.

This paper examines the impact of embedded options on the valuation of convertible debt in complex capital structures and characterizes the optimal conversion strategy. Convertible debt is a security that may, at the holder's option, be exchanged for a specified number of shares of newly issued common stock. Conversion has two offsetting effects: (1) it dilutes the fraction of the firm controlled by existing stockholders and (2) it reduces the

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[1]For example, the August 28, 1989 edition of *Value Line Convertibles* lists 418 firms with convertible debt. Fifty-two of these firms have multiple issues of convertible debt outstanding (43 firms have two issues and 9 firms have 3 issues).

[2]Capital structures are considered to be complex when the firm has a capital structure that is comprised of more than one type of security with an embedded option.

[3]See Black and Scholes (1973), Merton (1974), Ingersoll (1977), Brennan and Schwartz (1977), and Cox and Rubinstein (1985).

level of fixed claims. Since neither effect alters the firm's asset structure there is no incentive to convert prior to maturity. Conversion simply reallocates future cash flows among securityholders.

To compare the strategic implications for convertible debt with those for warrant conversion, it is useful to examine warrants more closely. A warrant holder converts by paying the exercise price. Since the firm must reinvest these exercise proceeds, warrant exercise is like a new equity issue that increases firm size and changes the firm's asset structure. The warrant valuation literature has shown that, if shareholders follow particular reinvestment policies for the exercise proceeds, sequential conversion strategies can increase warrant value. A conversion strategy is defined as sequential if it involves conversion of a portion of the outstanding securities prior to maturity.

A drawback of the papers examining sequential conversion strategies is their assumption that shareholders view these reinvestment policies as value maximizing. However, such policies fail on this dimension. In fact, Spatt and Sterbenz (1988) examine this issue and demonstrate that any gains accruing to warrantholders are derived at the expense of shareholders. They argue that, since current shareholders control reinvestment, the optimal reinvestment policy eliminates the incentive to convert sequentially. To illustrate their point, Spatt and Sterbenz (1988) provide several examples of reinvestment policies in which this incentive is eliminated. Hence, in equilibrium, warrants must be priced as if conversion occurs at maturity.[4]

By contrast, the optimal conversion strategy for convertible debt in complex capital structures has a strategic dimension that differs significantly from that described for warrants. Whether a conversion option is in-the-money at maturity depends on the conversion decisions of *all* bond classes. This paper shows that the payoffs for a given class of bondholders are not independent of the actions of other bond classes. As a result, the valuation equations developed for simple capital structures cannot be applied to individual bond classes in a complex capital structure.[5]

The remainder of this paper is organized as follows. Section 2 derives the optimal conversion strategies for different classes of convertible debt. Using

[4]In other words, assuming there are no distributions to shareholders that could trigger early exercise, conversion is effectively block-constrained. When securities are block-constrained, all outstanding securities are assumed to be exercised simultaneously. This result holds regardless of the distribution of securities across different owners. The distribution of securities across owners is an important issue in the warrant valuation literature because, if sequential conversion strategies increase a warrant's value, the largest gain is available to a monopoly holder of that security.

[5]Throughout this paper, a simple capital structure refers to one in which there is no more than one security outstanding with an embedded option. In addition, a security class refers to contractual differences among securities of a given type. For example, convertible debt with different face values and/or number of shares received upon conversion are considered to be in different classes.

these strategies, section 3 derives the valuation equations for convertible debt. The main results are summarized in section 4.

2. Optimal conversion strategies

This section derives the optimal conversion strategy for convertible debtholders when there exist multiple classes of convertible debt. The main issue is whether bondholders, within a given class, have an incentive to implement a sequential conversion strategy. An investor has such an incentive whenever sequential conversion increases the value of his underlying security. For example, Constantinides (1984) shows that competitive warrant holders engage in sequential exercise. He proves that the value of a warrant, if no one converts early, is less than its conversion value prior to maturity. This provides an incentive for early conversion. Since all investors recognize this, an equilibrium obtains wherein just enough warrants are converted so that this incentive is eliminated, i.e., when the warrant value equals its conversion value.[6]

Such strategies are common in the warrant valuation literature because sequential conversion affects the distribution of firm value on the expiration date. As a result, rational warrantholders convert sequentially if, given the firm's reinvestment policy for exercise proceeds, they can favorably influence the value of their claims. In contrast, convertible debt does not require the holder to pay anything upon conversion. Because bond conversion does not affect aggregate firm value or the distribution of firm value at maturity, bondholders within a given class have an incentive to convert sequentially only when they can expropriate wealth from other corporate claimants.

It is shown that bondholders do not have an incentive to engage in sequential conversion strategies. Further, it is never optimal to convert prior to maturity regardless of the number of classes of convertible debt. The intuition for this result follows from the observation that the aggregate market value of the firm's securities and the distribution of aggregate firm value are unaltered by conversion. Early conversion by any bondholder simply eliminates a valuable option.[7]

To demonstrate the dependence of conversion strategies across bond classes, capital markets are assumed to be perfect. In addition, consider a

[6]In the resultant equilibrium, the price of an unconverted warrant equal its conversion value. Constantinides (1984) also shows this is the price that would obtain if conversion is block-constrained.

[7]Given that conversion only occurs on maturity, by definition, conversion is block-constrained. Although this result appears analogous to Constantinides (1984), it is different in two respects: (1) conversion only occurs at expiration, and (2) the results are independent of the ownership distribution.

firm that does not make periodic dividend or interest payments and for which all debt has the same maturity date.[8] To illustrate this, let:

N ≡ common shares outstanding,
L ≡ number of bond classes, $l=1,\dots,L$,
F_l ≡ aggregate face value of bonds in class l,
$\Gamma(j)$ ≡ aggregate face value of convertible debt for classes $1,\dots,j$ $(j \leqq L)$. $\Gamma(L)$ is the aggregate face value across all classes,
$\Gamma^c(j)$ ≡ aggregate face value of convertible debt for classes $j+1,\dots,L$. By definition, $\Gamma(L)=\Gamma^c(j)+\Gamma(j)$ and $\Gamma^c(L)=0$,
m_l ≡ number of shares of common stock that are issued if class l bonds are converted,
$M(j)$ ≡ aggregate number of shares of common stock that are issued if bonds in classes $1,\dots,j$ are converted. $M(L)$ is the aggregate number of shares if all bondholders convert,
$\gamma(l,j+1)$ ≡ dilution factor for class l bonds if classes $1,\dots,j$ and l convert where $j \in \{j+1,\dots,L\}$,
$K(l,j+1)$ ≡ the effective exercise price for class l bonds if classes $1,\dots,j$ and l convert where $j \in \{j+1,\dots,L\}$,
ϕ_l ≡ proportion of the firm allocated to class l bondholders in default,
V_t ≡ aggregate firm value at time t.

To evaluate conversion strategies for different bond classes it is necessary to determine whether a conversion strategy for an individual class depends on the actions of the other classes. A given bond class's conversion policy is determined by deriving that class's effective exercise price and examining the conversion incentives that result from all possible conversion strategies. The optimal conversion policy is derived in two steps. First, the optimal conversion policy at maturity is established. Then it is shown that regardless of the ownership distribution, bondholders only convert at maturity.

Class l bonds will be converted at maturity when the proportion of the firm received after conversion exceeds the promised bond payment, i.e.

$$\gamma(l,j)[V_T-(\Gamma^c(j-1)-F_l)]>F_l.^9$$

This condition implies that the portion of the firm allocated to class l bondholders is reduced by the payments to nonconverting bondholders. The effective exercise price is

[8] We discuss the implications of these assumptions in section 3.

[9] Notice that this implicitly assumes that $j-1$ classes have converted and some class l, $l \in (j,\dots,L)$, is considering conversion. We present the effective exercise price in this manner because the conversion strategy implied by this assumption is consistent with the optimal conversion strategy.

$$K(l,j)=(F_l/m_l)(M(j-1)+N)+\Gamma^c(j-1). \quad (1)$$

If all classes convert, the effective exercise price for class l is

$$K(l,L)=(F_l/m_l)(M(L)+N). \quad (2)$$

Note that eq. (1) depends on class l through the ratio F_l/m_l. Also note that the terms $M(j-1)+N$ and $\Gamma^c(j-1)$ are constant for any class l where $l\in\{j+1,\ldots,L\}$. Given this observation, it is convenient to rank bond classes on the basis of their face value per converted share,

$$F_1/m_1<F_2/m_2<\cdots<F_L/m_L. \quad (3)$$

This characterization is without loss of generality because it does not restrict the face values or the number of shares across bond classes. In the results that follow, this ordering will display the intuitive property that the bondholders with lower face values per converted share have an incentive to convert at lower firm values.

The optimal conversion strategy is derived by establishing the effective exercise prices for different bond classes. Lemmas 1, 2 and 3 show that the effective exercise prices possess a sequential structure which allows for a straightforward characterization of the conversion strategy. This sequential structure is such that, if n groups convert at maturity, the n groups with the lowest face value per converted share have the lowest effective exercise prices. For example, Lemma 1 shows that, when only one class converts, class 1 bondholders have the lowest effective exercise price. Lemma 2 then establishes that $K(1,2)<K(1,1)<K(l,1)<K(l,2)$. In other words, if two classes convert, class 1 (l) is willing to convert at lower (higher) firm values than it would if it were the only group to convert. These lemmas are discussed below.

Lemma 1. If only one class converts at maturity, the bond with the lowest face value per converted share has the lowest effective exercise price,

$$K(1,1)<K(l,1) \quad \forall\, l\in\{2,\ldots,L\}.$$

Proof. The proofs of this and the following lemmas can be found in Appendix A.

Using the definitions of the effective exercise price in (1), Lemma 2 establishes the relative ordering of the effective exercise prices if only two groups convert.

Lemma 2. Given that only two groups convert, classes 1 *and* 2 *have the lowest* *effective exercise prices and are willing to convert at lower firm values than the remaining classes. This implies*

$$K(1,2)<K(1,1)<K(l,1)<K(l,2) \quad \forall\, l \in \{2,\dots,L\}$$

and

$$K(2,2)<K(l,2) \quad \forall\, l \in \{3,\dots,L\}.$$

The intuition for Lemma 2 follows from the observation that conversion has two offsetting effects: (1) the additional dilution from simultaneous conversion tends to increase the effective exercise price since the fraction of the firm allocated to each bond class is smaller $[m_j/(m_l+m_1+N)<m_j/(m_j+N),\ j=\{1,l\}]$, and (2) converting bondholders are not paid the face value. This latter result tends to reduce the effective exercise price. For the first class, eliminating the larger principal payment is sufficient to offset the additional dilution since, on a per-converted share basis, the face value of the second class is larger than the first. In contrast, the opposite holds for the second class.[10]

Lemma 3 generalizes this result for an arbitrary number of converting classes. In deriving Lemma 3, note that, if two bond classes convert, bond class 1's effective exercise price decreases while the effective exercise prices for the other classes increase.

Lemma 3. If n groups convert at maturity, the n groups with the lowest face value per converted share have the lowest effective exercise price,

$$K(1,n)<\cdots<K(n-1,n)<K(n-1,n-1)$$

$$<K(n,n)<K(l,n) \quad \forall\, l \in \{n+1,\dots,L\}.$$

Given Lemmas 1, 2, and 3, it is now straightforward to establish Proposition 1.

Proposition 1. For a capital structure comprised of common stock and L classes of European convertible debt, the optimal conversion policy for bond class l is to convert whenever V_T exceeds $K(l,l)$ at maturity.

[10]The structure of the effective exercise prices in Lemma 2 is sufficient to eliminate the possibility of a conversion conflict upon expiration. To illustrate this point, consider a capital structure comprised of equity and two classes of convertible debt. A conversion conflict is said to exist whenever both classes would convert individually if they were the only class to convert $(V_T-K(l,1)>0)$; yet neither would convert if both classes converted $(V_T-K(l,2)<0)$. Such a situation could occur if it is possible for $K(2,1)<K(1,2)$ when $K(1,1)<K(1,2)$. In this situation, the pure strategy Nash equilibrium is for neither class to convert (the conversion conflict is a prisoner's dilemma problem).

The proof of this and the next proposition can be found in Appendix A.

It is now shown that, regardless of the ownership distribution, bondholders do not have an incentive to convert early. As a consequence, the conversion strategy defined in Proposition 1 continues to be optimal even when conversion is permitted anytime over the life of the bond.

Proposition 2. For a capital structure comprised of common stock and L classes of convertible debt that may be converted any time at or before maturity, it will never be optimal to convert debt except at maturity, regardless of the ownership distribution. Thus, the optimal conversion policy for bond class l is to convert anytime V_T exceeds $K(l,l)$ at maturity.

3. Valuation of convertible debt in complex capital structures

This section examines the impact of the optimal conversion policy on convertible debt valuation. The previous section derived the optimal conversion policy where it was shown that conversion only occurs at maturity when there are no dividend payments. This result affects the conversion indifference point for bondholders and has a direct impact on the payoffs at maturity. Using the conversion results of section 2, closed-form solutions for convertible debt are derived using option pricing methods, in a world in which the Modigliani–Miller (1958) theorems obtain.[11]

To present the results as clearly as possible, the assumptions of section 2 are summarized below:

(A1) Trading occurs continuously in perfect and frictionless capital markets with no taxes, transactions costs or informational asymmetries. Investors act as price takers.
(A2) The firm's capital structure is comprised of common stock and L classes of non-callable convertible debt.
(A3) All classes of convertible debt expire on date T and there are no periodic interest payments.
(A4) No dividends are paid on common stock.

The first assumption is standard and is used to illustrate that the conversion results do not depend on market imperfections. The remaining assumptions define the specific capital structure we consider in this paper. They are designed to simplify the analysis so that the optimal bond conversion strategies can be presented in as clear a manner as possible.

The assumption that bonds are not callable [Assumption (A2)] allows us to abstract from the possibility of a 'forced' early conversion. This does not

[11]Capital structure irrelevance follows directly from Assumption (A1) and the observation in section 2 that bond conversion does not affect a firm's asset structure. Merton (1974) discusses this issue and his results are directly applicable.

affect our basic result that individual bond classes must consider the conversion decisions of other bond classes. To see this note that forced conversion is equivalent to 'voluntary' early conversion, i.e., both actions remove a valuable option. Since neither action affects the distribution of future cash flows, they leave the firm's asset structure intact, which implies that the results of Proposition 2 apply. In fact, forced conversion only makes it more likely that bond classes with lower effective exercise prices convert at maturity.

Assumption (A3) eliminates the incentive to convert early and capture a dividend (this assumption eliminates the possibility of early exercise that is associated with an 'American' call option). In addition, Assumptions (A3) and (A4) ensure that the firm is not required to make payments that have the potential to change the firm's asset structure prior to maturity. Taken together, these assumptions allow us to maintain the assertion that financing policy does not influence investment policy. That is, because there are no interim payments to bondholders or shareholders prior to maturity, the firm does not need to raise capital in financial markets to maintain the optimal (given) investment policy.

To derive solutions in closed-form, it is also necessary to make two additional assumptions that specify the stochastic properties of the firm. They are:

(A5) Firm value follows the lognormal diffusion process

$$\mathrm{d}V_t = \mu V_t \mathrm{d}t + \sigma V_t \mathrm{d}z,$$

where μ is the instantaneous expected rate of return on the firm, σ^2 is the instantaneous variance of the return on the firm, and z is a standard Weiner process.

(A6) The term structure of interest rates is 'flat' and nonstochastic. That is, the price of a riskless promised bond that promises a payment of one dollar at time τ in the future is $P(\tau) = \mathrm{e}^{-r\tau}$ where r is the instantaneous riskless rate of interest.

Because Assumptions (A1) through (A4) are sufficient to ensure that the firm's investment decisions and financing decisions are independent, Assumption (A5) is a convenient way to represent changes in firm value. Assumption (A6) is made to keep the analysis tractable (it is sufficient for closed-form solutions).

Given these assumptions, the price of convertible debt for class l, $C^l(V_t, t)$, must satisfy the stochastic differential equation

$$\tfrac{1}{2}\sigma^2 V^2 C^l_{vv} + rVC^l_v - rC^l + C^l_t = 0 \tag{4}$$

subject to the appropriate boundary conditions. The following two boundary conditions hold for all bond classes. The first condition is

$$C^l(0,t)=0 \tag{5}$$

and is satisfied by limited liability for financial securities. The second condition is a simple no-arbitrage requirement, i.e.

$$C^l(V_t,t)\leqq V_t. \tag{6}$$

Since convertible debt is only converted upon expiration at date T, the remaining boundary condition is the payoff at maturity. This condition must reflect the conversion decisions of class l and the effects of conversion by other classes. The payoff at maturity for class l is defined as

$$C^l(V_T,T)=\begin{cases}\phi_l V_T & V_T\leqq \Gamma(L),\\ F_l & \Gamma(L)<V_T<K(l,l),\\ \gamma(l,j)[V_T-\Gamma^c(j)] & K(j,j)\leqq V_T<K(j+1,j+1),\\ & \quad j=\{l,\dots,L-1\},\\ \gamma(l,L)V_T & V_T>K(L,L).\end{cases} \tag{7}$$

The payoff at maturity in (7) is equivalent to a portfolio that is comprised of ϕ_l units of a pure discount bond, a long position in a call option with exercise price $K(l,l)$, and short positions in $L-l-1$ call options with exercise prices $K(j,j)$ where $j=\{l+1,\dots,L\}$, i.e.

$$C^l(V_T,T)=\phi_l\min(V_T,\Gamma(L))+\gamma(l,l)\max[0,V_T-K(l,l)]$$
$$-\sum_{j=l}^{L-1}\gamma(j+1,j+1)\gamma(l,j)\max[0,V_T-K(j+1,j+1)]. \tag{8}$$

The short positions of $\gamma(j+1,j+1)\gamma(l,j)$ units of call options with exercise prices $K(j+1,j+1)$ reflect the additional dilution that occurs when bond classes with higher face values per converted share convert at maturity.

The price of class l debt is derived by solving eq. (4) subject to (5), (6) and (8). The solution is given by

$$C^l(V_t,t)=\mathrm{e}^{-r\tau}F_l\mathrm{N}(X(\Gamma(L))-\sigma\sqrt{\tau})-\phi_l V_t\mathrm{N}(-X(\Gamma(L)))$$

$$+\gamma(l,l)[V_t\mathrm{N}(X(K(l,l)))-\mathrm{e}^{-r\tau}K(l,l)\mathrm{N}(X(K(l,l))-\sigma\sqrt{\tau})]$$

$$-\sum_{j=l}^{L-1}\gamma(j+1,j+1)\gamma(l,j)[V_t\mathrm{N}(X(K(j+1,j+1)))$$

$$-\mathrm{e}^{-r\tau}K(j+1,j+1)\mathrm{N}(X(j+1,j+1)-\sigma\sqrt{\tau})], \tag{9}$$

where

$$X(y)\equiv[\ln(V_t/\mathrm{e}^{-r\tau}y)+\tfrac{1}{2}\sigma^2\tau]/\sigma\sqrt{\tau},$$

τ is the time to maturity ($\tau=T-t$), and $\mathrm{N}(\cdot)$ denotes the cumulative standard normal density function.

3.1. Numerical example

In this subsection, we illustrate the results of this section with a numerical example. Suppose a firm has a capital structure comprised of 2,000 shares of common stock and two classes of convertible debt. The bonds in the first class have a face value of $50,000 and can be converted into 3,000 shares. The bonds in class 2 have a face value of $70,000 and can be converted into 1,500 shares. The effective exercise prices for classes 1 and 2 if only one class converts are $153,333 and $213,333, respectively. If both classes convert, the effective exercise prices for classes 1 and 2 are $108,333 and $303,333. Using eq. (8), fig. 1 illustrates the payoff at maturity for common stock and the two bond classes.

To value the firm's capital structure assume that the instantaneous standard deviation is 20%, and the time to maturity is one year. Using eq. (9), fig. 2 illustrates the market value of common stock and both classes as a function of firm value.

It is possible to illustrate some comparative static results using this example. For example, figs. 3 and 4 illustrate how convertible bonds respond to different standard deviation levels and times to maturity. In this example, class 2 bonds respond in an analogous manner to noncallable, convertible bonds in a simple capital structure. This allows us to consider class 2 debt as a benchmark case. Fig. 3 illustrates that class 1 debt is less responsive than class 2 debt to changes in the standard deviation of the underlying firm's instantaneous rate of return. This observation is interesting because the

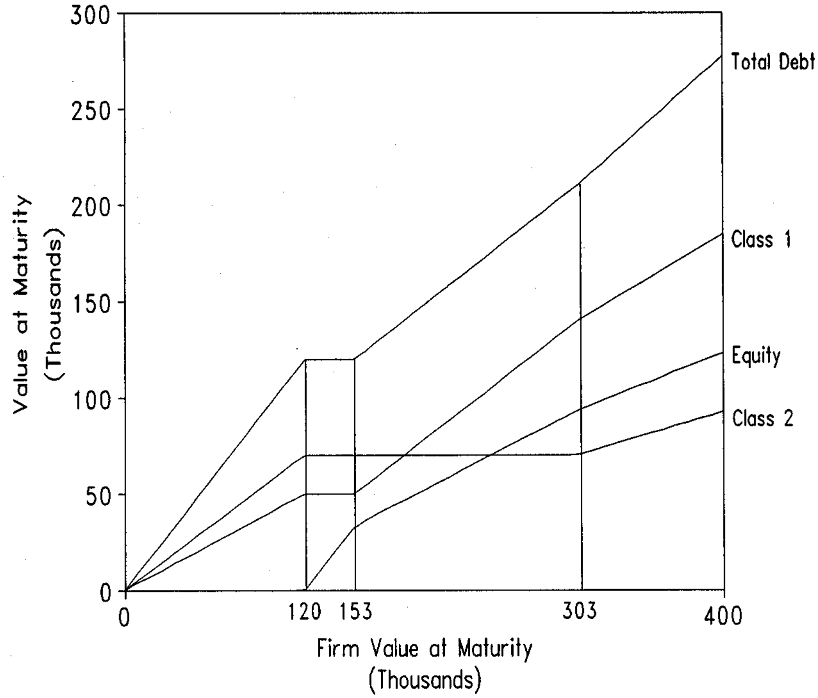


Fig. 1. Plots of the boundary condition at maturity for two classes of convertible debt and equity at different firm values. These plots were constructed with the following parameter values: $r=5\%$, standard deviation $=20\%$, $n=2{,}000$, time to maturity $=1$, $F_1=\$50{,}000$, $m_1=3{,}000$, $F_2=\$70{,}000$, and $m_2=1{,}500$.

payoff on class 2 bonds is equivalent to the payoff on a convertible bond in a simple capital structure.[12] Fig. 4 illustrates that class 1 bonds are also less responsive to time to maturity than class 2 bonds.

4. Conclusion

Early studies of convertible securities have argued that sequential conversion strategies can increase value for certain ownership distributions and investment policies. In this paper, the impact of conversion strategies on the value of convertible debt is examined for capital structures comprised of multiple classes of convertible debt. First, the optimal conversion policy is derived. Second, we show that, in general, the conversion decisions for a

[12] Brennan and Schwartz (1988) note that risk insensitivity may explain why convertible debt is a popular financing vehicle. They argue that '(risk insensitivity) allows (convertible bonds) to be issued on terms that look fair to management, even when the market rates the risk of the issuer higher than does management of the issuing company.'

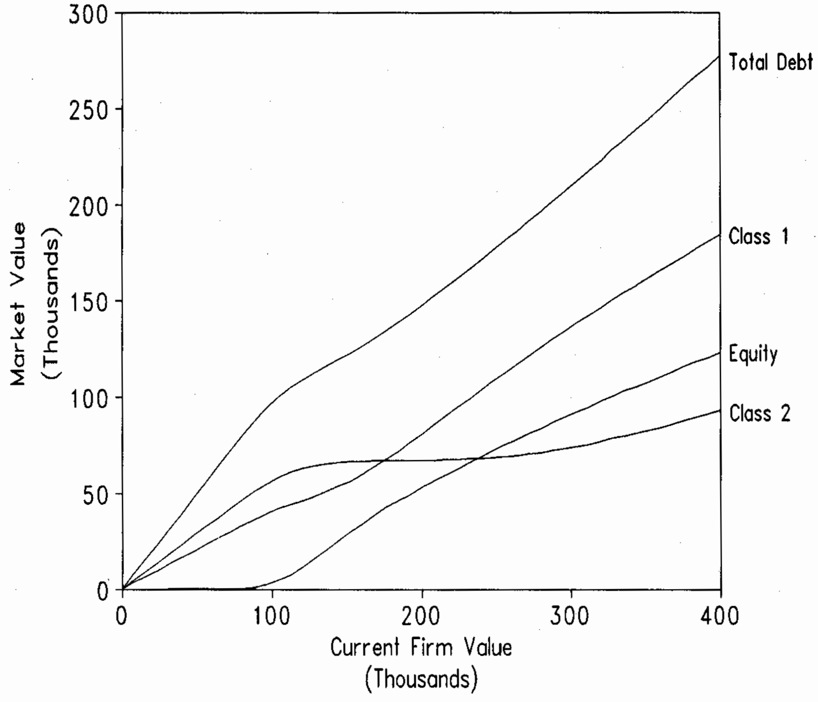


Fig. 2. Plots of the current market value for two classes of convertible debt and equity at different firm values. These plots were constructed with the following parameter values: $r=5\%$, standard deviation $=20\%$, $n=2{,}000$, time to maturity $=1$, $F_1=\$50{,}000$, $m_1=3{,}000$, $F_2=\$70{,}000$, and $m_2=1{,}500$.

given class are affected by the decisions of other convertible debt classes. Finally, we establish that convertible debt in complex capital structures is priced as if conversion occurs at maturity, and closed-form solutions for bond prices are derived.

In this paper, a number of simplifying assumptions are made so that we can draw sharp conclusions regarding the optimal conversion policy. For example, we assume that convertible bonds are noncallable, they do not make periodic interest payments, and the firm does not pay dividends. These features which are present in virtually all convertible debt will generally result in early conversion. We do not explicitly model them because they complicate the analysis without affecting our main result that convertible bondholders consider the conversion decisions of other security holders. Further research in this area is needed to determine if these results generalize to even more realistic settings. For example, it would be interesting to consider alternative capital structures such as one comprised of a combination of multiple classes of warrants and convertible debt.

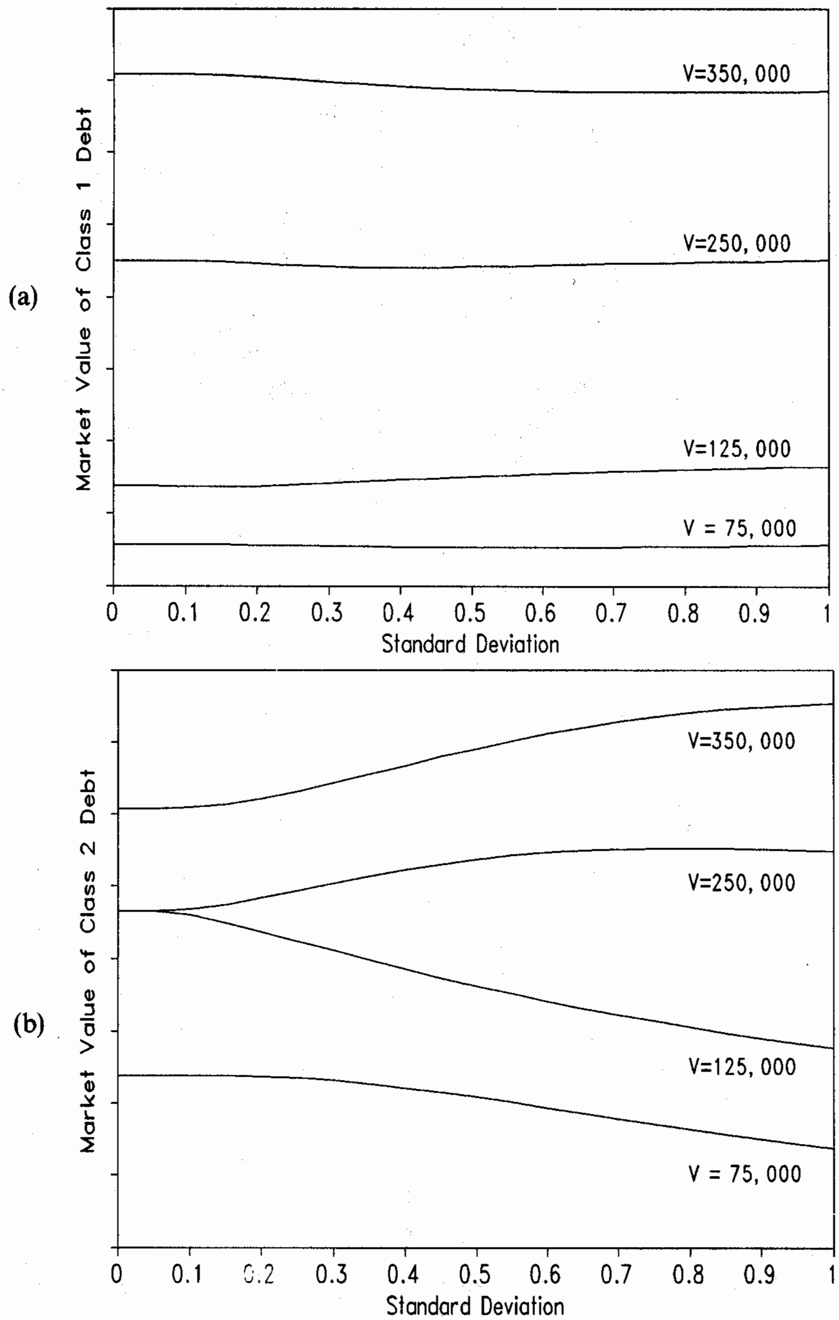
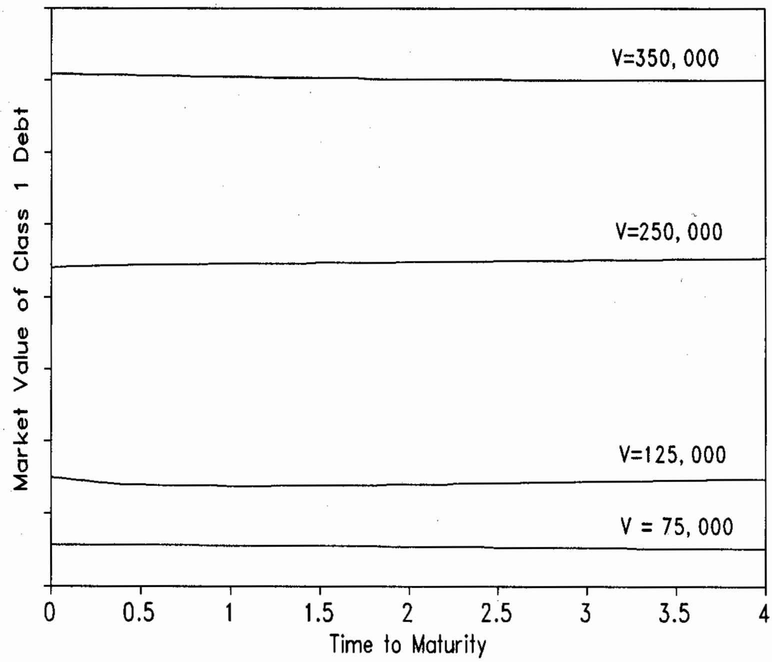


Fig. 3. Plots of the current market value for class 1 and 2 convertible debt at different standard deviation levels. Each line represents a different firm value ($75,000, $125,000, $250,000, $350,000). These plots were constructed with the following parameter values: $r=5\%$, standard deviation$=20\%$, $n=2{,}000$, time to maturity$=1$, $F_1=\$50{,}000$, $m_1=3{,}000$, $F_2=\$70{,}000$, and $m_2=1{,}500$.

(a)



(b)

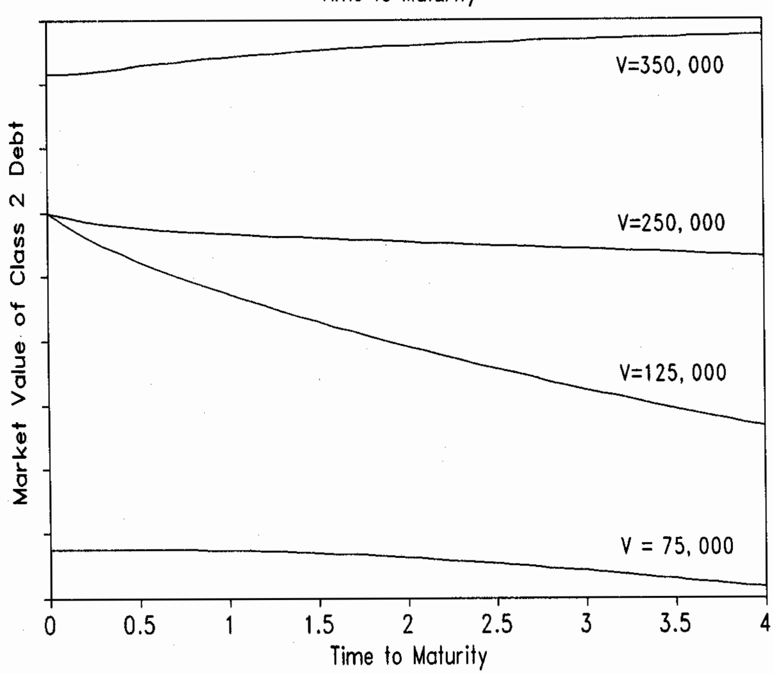


Fig. 4. Plots of the current market value for class 1 and 2 convertible debt at different times to maturity. Each line represents a different firm value ($75,000, $125,000 $250,000, $350,000). These plots were constructed with the following parameter values: $r=5\%$, standard deviation$=20\%$, $n=2{,}000$, time to maturity$=1$, $F_1=\$50{,}000$, $m_1=3{,}000$, $F_2=\$70{,}000$, and $m_2=1{,}500$.

Appendix A

This appendix contains the details of the proofs to Lemmas 1, 2, and 3 and Propositions 1 and 2.

Proof of Lemma 1. Follows directly from the definition of $K(L,1)$ and eq. (3). □

Proof of Lemma 2. First show that $K(1,2)<K(1,1)$, $\forall l\in\{2,\dots,L\}$. Given that two bond classes convert at maturity, the effective exercise price for a class 1 bondholder is

$$\begin{aligned}K(1,2,l)&=(F_1/m_1)(m_1+m_l+N)+\Gamma(l)-F_1-F_l\\&=K(1,1)+m_l(F_1/m_1-F_l/m_l)\\&<K(1,1),\end{aligned}$$

where $K(j,n,l)$ denotes the effective exercise price for bond class j ($j\leqq n-1$) when classes $1,\dots,n-1$ and l convert. The inequality is an implication of (3).

Next, show that $K(1,1)<K(l,2)$. Given that two bond classes convert at maturity, the effective exercise price for a class l bondholder is

$$\begin{aligned}K(l,2,l)&=(F_l/m_l)(m_1+m_l+N)+\Gamma(L)-F_1-F_l\\&=K(l,2)+m_1(F_l/m_l-F_1/m_1)\\&>K(l,2).\end{aligned}$$

The inequality is an implication of (3).

To show that $K(2,2)<K(l,2)$, $\forall l\in\{2,\dots,L\}$ it is sufficient to note $K(l,2)$ is increasing in F_l/m_l. □

Proof of Lemma 3. Following the lines of the proof to Lemma 2, show that

$$K(1,n)<\cdots<K(n-1,n)<K(n-1,n-1),\quad \forall l\in\{n,\dots,L\}.$$

Given that bond classes $1,\dots,n-1$ and l convert at maturity (a total of n classes), the effective exercise price for bond class j where $j\in\{1,\dots,n-1\}$ is

$$K(j,n,l)=(F_j/m_j)(M(n-1)+m_l+N)+\Gamma(L)-\Gamma(n-1)-F_l \tag{A.1}$$

or

$$K(j,n,l)=K(j,n-1)+M_l(F_j/m_j-F_l/m_l),$$

where $K(j,n,l)$ denotes the effective exercise price for bond class j ($j \leqq n-1$) when classes $1,\dots,n-1$ and l convert. Eq. (A.1) is increasing in (F_j/m_j). Hence, $K(1,n)<\cdots<K(n-1,n)$ is proven. To show that $K(n-1,n)<K(n-1,n-1)$, evaluate (A.1) at $j=n-1$.

$$K(n-1,n,l)=K(n-1,n-1)+m_l(F_{n-1}/m_{n-1}-F_l/m_l)<K(n-1,n-1). \tag{A.2}$$

The inequality is an implication of (3).

Next, it is shown that

$$K(n-1,n-1)<K(n,n)<K(l,n), \quad \forall l \in \{n,\dots,L\}.$$

Given that classes $1,\dots,n-1$ and l convert, the effective exercise price for class l is

$$\begin{aligned} K(l,n,l) &= (F_l/m_l)(M(n-1)+m_l+N)+\Gamma(L)-\Gamma(n-1)-F_l \\ &= (F_l/m_l)(M(N-1)+N)+\Gamma(L)-\Gamma(n-1) \\ &> K(n-1,n-1). \end{aligned} \tag{A.3}$$

The inequality follows because $F_l/m_l>F_{n-1}/m_{n-1}$. Since (A.3) is increasing in F_l/m_l, $K(n,n)<K(l,n)$ is an implication of eq. (3). □

Proof of Proposition 1. The optimal conversion policy is determined inductively using Lemmas 1, 2, and 3.

Lemma 1 establishes that bond class 1 is willing to convert at lower firm values than any other bond class when only one bond class converts.

Lemma 2 shows that if two classes convert, class 1 (the class with the strongest incentive to convert if only one class converts) is willing to convert at lower firm values than any other group. Group 1's effective exercise price if two classes convert is lower than if only one class converts. By contrast, the effective exercise prices for the remaining $L-1$ classes are higher if two classes convert than if each class converts by itself. Among these, class 2 has the lowest effective exercise price. Hence, if only two classes convert, the converting classes are 1 and 2.

Lemma 3 then establishes the induction. In n classes convert, the effective exercise prices for the first $n-1$ classes (those with the strongest incentive to convert if only $n-1$ classes convert) are willing to convert at lower firm values than any other class. The first $n-1$ classes effective exercise prices if n classes convert is lower than if only $n-1$ classes convert. By contrast, the effective exercise prices for the remaining $L-n$ classes are higher if n classes convert than if only $n-1$ classes convert. Hence, if n classes convert, the first n classes are the ones that convert.

As a consequence, the optimal conversion policy is for class n to convert whenever $V_T \geqq K(n,n)$. □

Proof of Proposition 2. This proof proceeds by showing that a monopoly bondholder's optimal conversion strategy is to convert at maturity. Next, we establish that this result is independent of the ownership distribution.

Consider the case of the monopolist first. Since the monopolist bondholder's optimal conversion strategy is to maximize the value of convertible bonds at each instant in time, it is never optimal to convert if a bond's market value exceeds its conversion value. Brennan and Schwartz (1977) establish in their Lemma 1 (p. 1702) that, if this condition did not hold, the return on the bond would display first degree stochastic dominance over the return on the underlying stock. As Brennan and Schwartz (1977) note: 'Therefore, the bond will always sell above conversion value, and the investor will never find it optimal to convert.' The intuition for the stochastic dominance argument is that an option is worth more alive than dead. The reason this argument works is that conversion does not affect aggregate firm value. Since there are no cash flow effects, firm value at maturity is unaffected by the bondholder's conversion strategy.

Although Brennan and Schwartz (1977) establish this result for a single class of convertible debt, it also holds for multiple issues of convertible debt. To see this note that, by applying a continuity argument to Lemma 3, partial conversion of a given class's bonds before maturity, reduces the effective exercise prices for those classes that would have converted at maturity anyway. As a consequence, early conversion simply eliminates a valuable option.

Next, we establish that the result is independent of the ownership distribution. A monopolist bondholder has complete flexibility in implementing a conversion strategy. Since his optimal strategy is to hold all bonds to maturity, regardless of their class, the same must hold for any other ownership distribution since the strategy that is optimal for the monopolist is feasible under any arbitrary ownership distribution. □

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