by Robert E. Whaley

On Valuing American Futures Options

More than 20 different futures option contracts currently trade on U.S. exchanges. These "American" options are exercisable at any time up to and including the expiration day. In pricing them, however, investment managers have been forced to rely on principles developed for "European" options, which may be exercised only at expiration. This practice can be misleading, because the early exercise privilege of American futures options has a significant effect on pricing.

The early exercise premium of American futures options affects two types of pricing relations. The first type are those relations developed by assuming the market is free from costless arbitrage opportunities. These relations are often termed "rational option pricing restrictions," and an important relation within this category is the put-call parity relation, which simultaneously links the prices of the put and the call in the futures option market with the price of the underlying futures contract.

The second and perhaps most important type of option pricing relations affected by early exercise are valuation equations. Valuation equations require an additional assumption about the futures price distribution; the most commonly used assumption is a lognormal distribution. The widest known model for pricing futures option contracts is the Black model, but it was developed for European futures options and thereby ignores the value of the early exercise feature of the American options. An intuitively appealing approximation method based on the American futures option valuation equation is very accurate and computationally inexpensive.

OPTIONS ON FUTURES CONTRACTS were introduced in the U.S. only four years ago.[1] Now more than 20 different futures option contracts are actively traded on every major futures exchange. The Chicago Mercantile Exchange (CME) trades options based on the S&P 500, the West German mark, the British pound, the Swiss franc, Eurodollars, live cattle and live hogs. The Chicago Board of Trade (CBT) has U.S. Treasury bond, U.S. Treasury note, silver, corn and soybean futures options. The New York Futures Exchange (NYFE) has NYSE composite index equity futures options, and the Commodity Exchange (CMX) has gold and silver futures options. Even the smaller exchanges, such as the Kansas City Board of Trade (KC), the Minneapolis Grain Exchange (MPLS), the MidAmerica Commodity Exchange (MCE), the New York Cotton Exchange (CTN) and the Coffee, Sugar and Cocoa Exchange (CSCE) now maintain active secondary markets in futures option contracts.

The alacrity with which these new contingent claims have captured the attention of financial analysts and portfolio managers argues for a review and extension of the fundamentals of futures option valuation. In 1976, Black provided a framework for analyzing commodity futures options.[2] His work was explicitly directed at pricing "European" futures options—that is, options that may be exercised only at expiration. The "American" options currently trading,

1. Footnotes appear at end of article.

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This research was supported by the Institute for Financial Research, University of Alberta. Comments and suggestions by Warren Bailey, Bruce Cooil, Theodore E. Day, Thomas S.Y. Ho and H.R. Stoll on an earlier version of this paper are gratefully acknowledged.

Table I Transactions for Futures-Forward Contract Price Equivalence

Position	Foward Portfolio Initial Value	Foward Portfolio Terminal Value	Futures Portfolio Initial Value	Futures Portfolio Terminal Value
Long Bonds	fe^{-rT}	f	Fe^{-rT}	F
Long Forward Contract	0	$S_T - f$		
Long "Rollover" Futures Position			0	$S_T - F$
Net Value	fe^{-rT}	S_T	Fe^{-rT}	S_T

however, may be exercised at any time up to and including the expiration day.

Although much has been written about futures options since the first contract applications were placed before the Commodity Futures Trading Commission (CFTC) in the early 1980s, most of the work has deferred to Black's European futures option pricing results.[3] Not until very recently has substantive progress been made in understanding the value of the early exercise privilege of American futures options and in providing more computationally efficient methods for pricing American options.[4]

This article clarifies the principles and intuition underlying European futures option pricing, extends these principles and intuition to American futures option pricing, and provides a simple and computationally efficient method for pricing American futures options. Most of the published work on futures option pricing actually represents work on forward option pricing. While the distinction between a forward and a futures contract is not particularly important in pricing European options, it is of critical importance in pricing their American counterparts. We thus begin with a short discussion of the difference between forward and futures contracts.

Futures Vs. Forward Contracts

Before considering futures option pricing relations, it is useful to distinguish between a futures contract and a forward contract. A *forward contract* is an agreement to deliver the underlying asset at a future time T at a price specified today. Payment for the asset takes place at time T, and no intermediate payments are made. A *futures contract* is similar to a forward contract, except that intermediate cash payments (receipts) are made as losses (profits) are incurred when the futures position is marked to market each day during the contract's life. These profits and losses accumulate interest during the contract's life so that, in general, the terminal value of a long futures contract position differs from that of a long forward contract position.

Although the terminal values of the two contract positions differ, the price of a forward contract, f, will equal the price of a futures contract, F, if the gains and losses on the futures position accumulate at a known riskless rate of interest.[5] To see this, consider two portfolios in a market that affords no costless arbitrage opportunities. The first portfolio consists of a long position of fe^{-rT} riskless bonds and a long forward contract.[6] The second consists of a long position of Fe^{-rT} riskless bonds and a long "rollover" futures position, where $e^{-r(T-1)}$ futures contracts are purchased the first day, $e^{-r(T-2)}$ the second day, $e^{-r(T-3)}$ the third day, and so on.[7] The number of futures contracts purchased increases by a factor of e^r each day, so that on the last day exactly one long futures contract is held.[8]

As Table I shows, the value of each portfolio position at time T equals the underlying commodity price, S_T. This being the case, the initial values of the portfolios must also be the same; otherwise arbitrageurs would step in to earn costless profits. The price of the forward contract must equal the price of the futures contract.

Futures Options Vs. Stock Options

A *futures option contract* is similar to an option on a stock, in the sense that it provides its holder with the right to buy or sell the underlying security at the exercise price of the option. Unlike a stock option, however, a futures option does not involve a cash exchange in the amount of the exercise price when the futures option is exercised.

Upon exercise, a futures option holder merely acquires a long or short futures position with a futures price equal to the exercise price of the

Table II Arbitrage Transactions for Put-Call Parity of European Futures Options

Position	Initial Value	Terminal Value $F_T < X$	Terminal Value $F_T \geq X$
Long "Rollover" Futures	0	$F_T - F$	$F_T - F$
Long Put Option	$-p(F,T;X)$	$X - F_T$	0
Short Call Option	$+c(F,T;X)$	0	$-(F_T - X)$
Long $(F - X)e^{-rT}$ Bonds	$-(F - X)e^{-rT}$	$F - X$	$F - X$
Net Value	$c(F,T;X) - p(F,T;X) - (F - X)e^{-rT}$	0	0

option. When the futures contract is marked to market at the close of the day's trading, the option holder is free to withdraw in cash an amount equal to the futures price less the exercise price in the case of a call or the exercise price less the futures price in the case of a put. Exercising a futures option is thus tantamount to receiving in cash the exercisable value of the option.

Put-Call Parity

In stock option markets, arbitrageurs and floor traders hold the prices of the put, the call and the underlying stock in a certain configuration by engaging in conversion and reverse conversion trading strategies.[9] The essential feature of these strategies is the recognition that the payoff contingencies posed by a long call position may be duplicated by a portfolio consisting of a long stock, a long put and some riskless borrowing. Therefore, if the price of the call exceeds the sum of the prices of the portfolio's securities, the arbitrageur can earn a costless profit by selling the call and buying the portfolio (i.e, by enacting a conversion). Conversely, if the price of the call is less than the sum of the prices of the portfolio's securities, a costless profit can be earned by buying the call and selling the portfolio (i.e., enacting a reverse conversion).[10]

In futures option markets, the same principles apply. Arbitrageurs who continually search for and take advantage of costless profit opportunities force particular configurations of futures option prices. These configurations are expressed in the form of pricing relations that have come to be known as "put-call parity theorems."

In addition to the assumption that the marketplace is free of costless arbitrage opportunities, this analysis requires that markets are frictionless (i.e., there are no transaction or similar costs) and that individuals can borrow or lend risklessly at a continuously compounded rate of interest rate, r. Neither of these assumptions is particularly restrictive; arbitrageurs pay minimal transaction costs, and the riskless rate is fairly constant over short intervals of time.

European Futures Options

The put-call parity relation for European futures options is as follows:[11]

$$C(F,T;X) - p(F,T;X) = (F - X)e^{-rT}, \quad (1)$$

where $c(F,T;X)$ and $p(F,T;X)$ are the prices of a European call and put, respectively, with exercise price X and time to expiration T.

This relation is driven by a conversion arbitrage portfolio consisting of (a) a long position in the futures contract, (b) a long position in the European put, (c) a short position in the call, and (d) a long position of $(F - X)e^{-rT}$ bonds. The long futures position is identical to that used above. On the first day, $e^{-r(T-1)}$ contracts are purchased, on the second $e^{-r(T-2)}$, $e^{-r(T-3)}$ on the third, and so on. Table II gives the initial and terminal values of this portfolio. Because the terminal value of the portfolio is certain to be zero, the initial value must also equal zero, and therefore Equation (1) must hold.

Note that this conversion arbitrage strategy calls for daily revision of the futures position. Earlier researchers using a static "buy-and-hold" futures contract position erroneously treated the futures contract as a forward contract in their proofs of European put-call parity and realized the correct pricing relation only because the options were European in nature.[12] If this approach were used in deriving American put-call parity, the resulting relation would be misspecified.

American Futures Options

The put-call parity relation for American futures options, like that for American spot options, is represented by two inequalities, as follows:[13]

$$Fe^{-rT} - X \le C(F,T;X) - P(F,T;X) \le F - Xe^{-rT}, \quad (2)$$

where C(F,T;X) and P(F,T;X) are the prices of an American call and put, respectively, with exercise price X and time to expiration T.

This relation is driven by two separate sets of arbitrage transactions. The left-hand-side of Equation (2) requires a reverse conversion arbitrage portfolio consisting of (a) a long position in the call, (b) a short position in the put, (c) a short position in the futures, and (d) a short position of $Fe^{-rT} - X$ bonds. Here, the short futures position is just the opposite of the rollover strategy applied to derive European put-call parity. That is, $e^{-r(T-1)}$ futures contracts are sold the first day, $e^{-r(T-2)}$ the second day, $e^{-r(T-3)}$ the third day, and so on. Table III presents the initial, intermediate and terminal values of the overall portfolio.

As Table III shows, if the put option is not exercised early, the terminal value of the portfolio is certain to be positive. The only uncertainty faced by the portfolio holder rests with the short position in the put, because it may be exercised against the portfolio holder at any time during the option's life. If the put option is exercised early, however, the payment of the exercise price is more than covered by riskless borrowing, and the assumed long futures position is less than fully offset by the short futures position established at the outset. The net value of the portfolio at early exercise thus equals the sum of three components, each of which has a value at least equal to zero. With the intermediate and terminal values of this portfolio being nonnegative, the initial value must be nonpositive, so the left-hand-side inequality of Equation (2) must hold.

To understand the right-hand inequality of Equation (2), consider a conversion arbitrage portfolio consisting of (a) a long position in the put, (b) a long position in the futures, (c) a short position in the futures and (d) a long position of $F - Xe^{-rT}$ bonds. Here the long futures position differs slightly from the rollover strategy described earlier. The investor purchases e^r futures on the first day, e^{2r} the second day, e^{3r} the third day, and so on. Each day the number of futures contracts bought increases by a factor of e^r, and on the last day e^{rT} contracts are held. Table IV gives the initial, intermediate and terminal values of these portfolios. As the intermediate and terminal values of the portfolio are certain to be nonnegative, the initial value must be nonpositive and the right-hand side of Equation (2) must hold.

In summary, certain futures option pricing relations are dictated by the absence of costless arbitrage opportunities in an efficiently operating marketplace. For the American-style futures options trading in the U.S. today, the put-call parity condition of Equation (2) represents one such relation. If it is violated in any of the existing futures option markets, either the reverse conversion arbitrage strategy depicted in Table III or the conversion arbitrage strategy depicted in Table IV may be enacted to earn a costless arbitrage profit. The relation does *not* depend on the nature of the commodity underlying the futures contract: It applies to agricultural futures option contracts, as well as those written on financial instruments, currencies and precious metals.

Valuation Equations

By far the more interesting pricing relations from a financial analyst's standpoint are valuation equations. They provide the guidance in the never-ending search to identify mispriced securities and to tailor the risk-return properties of contingent claims within a portfolio context.

Unlike the put-call parity relations, valuation equations require an assumption about the nature of the underlying futures price distribution. In option pricing theory, the most common assumption is that the price of the instrument underlying the option contract is lognormally distributed. This assumption is intuitively appealing, because the lowest price a security can attain is zero and the highest price is unlimited. The lognormal price distribution assumption is used to obtain the following futures option pricing results.

European Futures Options

As noted, Black derived the valuation equation for a European call option on futures contracts. If futures prices are lognormally distributed, and if a riskless hedge may be formed between the European call and its underlying futures contract, the value of a European call may be expressed as follows:[14]

$$c(F,T;X) = e^{-rT}[FN_1(d_1) - XN_1(d_2)], \quad (3)$$

where

Figure A European and American Call Option Prices as a Function of the Underlying Futures Contract Price

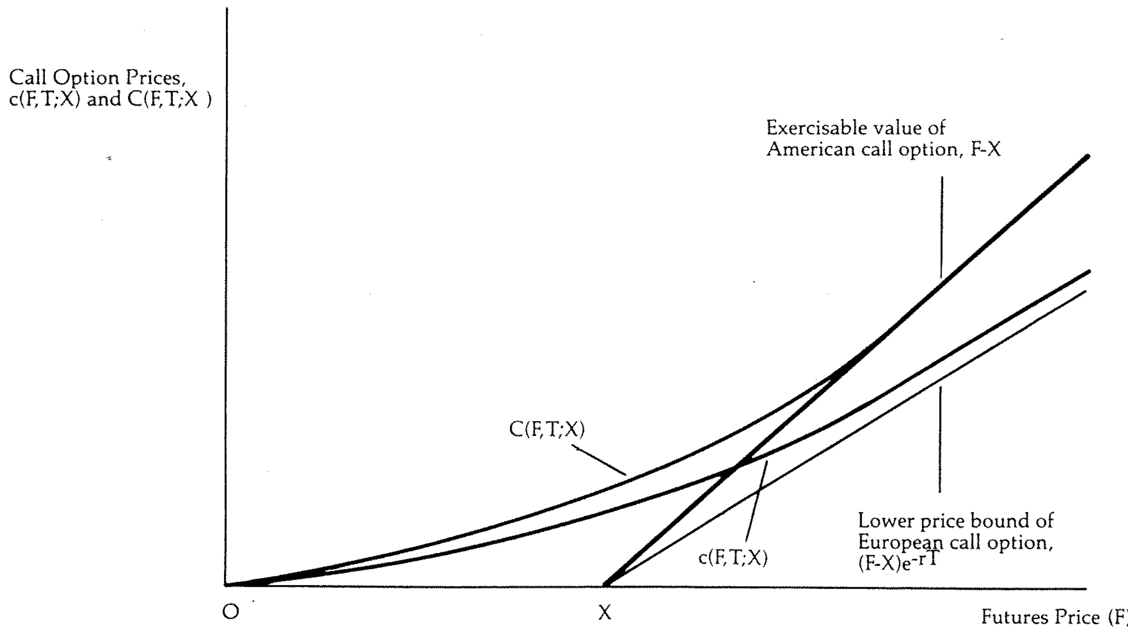

Figure B Call Option Price as a Function of Futures Price at the Early Exercise Instant t

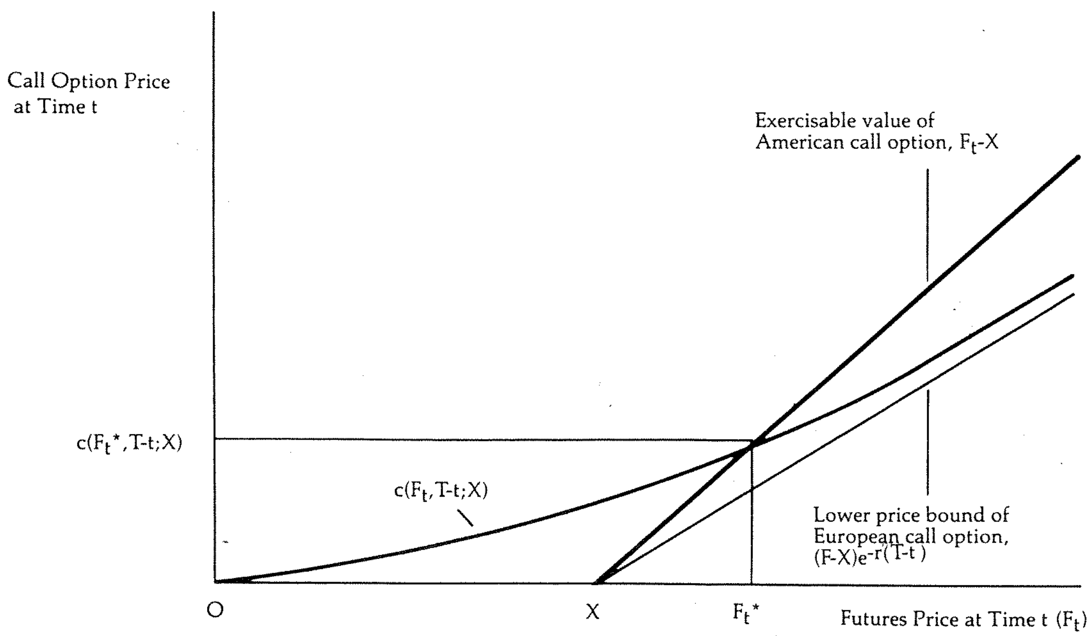

Table III Arbitrage Transactions for Put-Call Parity of American Futures Options, $Fe^{-rT} - X \leq C(F,T;X) - P(F,T;X)$

Position	Initial Value	Intermediate Value	Terminal Value $F_T < X$	Terminal Value $F_T \geq X$
Long Call Option	$-C(F,T;X)$	C_t	0	$F_T - X$
Short Put Option	$+P(F,T;X)$	$-(X - F_t)$	$-(X - F_T)$	0
Short Futures	0	$-(Ft - F)e^{-r(T-t)}$	$-(F_T - F)$	$-(F_T - F)$
Short $Fe^{-rT} - X$ Bonds	$+Fe^{-rT} - X$	$-[Fe^{-r(T-t)} - Xe^{rt}]$	$-(F - Xe^{rT})$	$-(F - Xe^{rT})$
Net Value	$Fe^{-rT} - X - [C(F,T;X) - P(F,T;X)]$	$C_t + F_t[1 - e^{-r(T-t)}] + X(e^{rT} - 1)$	$X(e^{rT} - 1)$	$X(e^{rT} - 1)$

Table IV Arbitrage Transactions for Put-Call Parity of American Futures Options, $C(F,T;X) - P(F,T;X) \leq F - Xe^{-rT}$

Position	Initial Value	Intermediate Value	Terminal Value $F_T < X$	Terminal Value $F_T \geq X$
Long Put Option	$-P(F,T;X)$	P_t	$X - F_T$	0
Long Futures	0	$(F_t - F)e^{rt}$	$(F_T - F)e^{rT}$	$(F_T - F)e^{rT}$
Short Call Option	$+C(F,T;X)$	$-(F_t - X)$	0	$-(F_T - X)$
Long $F - Xe^{-rT}$ Bonds	$-(F - Xe^{-rT})$	$Fe^{rt} - Xe^{-r(T-t)}$	$Fe^{rT} - X$	$Fe^{rT} - X$
Net Value	$C(F,T;X) - P(F,T;X) - (F - Xe^{-rT})$	$P_t + F_t(e^{rT} - 1) + X[1 - e^{-r(T-t)}]$	$F_t(e^{rT} - 1)$	$F_t(e^{rT} - 1)$

$$d_1 = [\ln(F/X) + 0.5\,\sigma^2 T]/\sigma\sqrt{T}, \text{ and}$$

$$d_2 = d_1 - \sigma\sqrt{T},$$

and where $c(F,T;X)$ is the price of a European call with exercise price X and time to expiration T. The current futures price is F, the riskless rate of interest is r, and the instantaneous standard deviation of the relative price changes in the futures contract is σ. The term $N_1(d)$ is the probability that a unit normally distributed random variable x will be less than or equal to d.

Equation (3) may seem without intuitive appeal, but it is not. It merely says that the current value of the call equals the present value of its expected value at expiration. At expiration time T, the futures option is worthless if it is out-of-the-money (i.e., if $F_T < X$) and it is worth $F_T - X$ if it is in-the-money (i.e., $F_T > X$).

The expected value of the call option at expiration is thus the expected difference between the futures price and the exercise price conditional upon the option being in-the-money times the probability that the call option will be in-the-money. This is precisely the meaning of the term $FN_1(d_1) - XN_1(d_2)$ in Equation (3). The term e^{-rT} is the appropriate discount factor by which the expected expiration value is brought back to the present.[15] The term $N_1(d_2)$ is the probability that the futures price will exceed the exercise price at the option's expiration.[16]

American Futures Options

Although the Black model is commonly used to price futures options, it may seriously understate the value of an American call option on a futures contract because it fails to account for the potential benefit of exercising the option early. Consider deep in-the-money call options. If Equation (3) is used to price the futures option, the call's value will be $e^{-rT}(F - X)$, because the values of $N(d_1)$ and $N(d_2)$ are approximately equal to one. The American call, however, can be exercised immediately for $F - X$, which is greater than the European call price by an amount equal to $(F - X)(1 - e^{-rT})$. In other words, the Black model underprices a deep in-the-money call by an amount equal to the present value of the interest that could be earned on the exercisable proceeds of the option if the option were exercised immediately.

In general, there is always some probability that a call option will go deep in-the-money, so the American call option price must include a premium that accounts for the potential benefit of early exercise. Figure A shows that the exercisable value of the American call option, $F - X$, always exceeds the lower price boundary of the corresponding European futures option, $(F - X)e^{-rT}$, so the American call may be worth more "dead" than "alive."[17] The difference betwen the curves $C(F,T;X)$ and $c(F,T;X)$ is the amount of the early exercise premium. The curve C(F,T;X) intersects the exercisable value of the American call at a futures price level where it is optimal to exercise the option immediately.

A technical explanation of the analytical valuation equation for an American call option on a futures contract is beyond the scope of this paper.[18] An intuitively appealing approximation method based upon the valuation equation is discussed below, however.

A Compound Valuation Approach

Consider the following sequence of "pseudo-American" call option prices. The first element of the sequence, C_1, is the price of a call option that can be exercised only at expiration; the second element, C_2, is the price of a call that can be exercised exactly one-half of the way through the call's life or at expiration; the third element, C_3, is the price of a call that can be exercised exactly one-third of the way through the option's life, two-thirds of the way through the option's life or at expiration, and so on. The value of each new call option introduced into the sequence has a greater value than the previous element, because it offers additional early exercise opportunity. If the sequence is continued indefinitely, the limiting value will be the price of a pseudo-American call with an infinite number of early exercise opportunities or, equivalently, the price of an American call option written on a futures contract.

The formula for the limiting value of the sequence has an infinite number of terms, hence is not a practical means of estimating the values of American call options written on futures contracts. All is not lost, however. The American call can be accurately priced by combining the first three elements of the sequence as follows:[19]

$$C(F,T;X) = 0.5\,C_1 - 4\,C_2 + 4.5\,C_3. \qquad (4)$$

The values of C_1, C_2 and C_3 are used to extrapolate the limiting value of the sequence.

The problem, then, becomes one of pricing the first three pseudo-American call options. The value of C_1 is easily computed using Equation (3), because this pseudo-American call option has no early exercise opportunities. As noted earlier, C_1 is simply the present value of the expected terminal value of the call conditional on the call finishing in-the-money times the probability that the call will finish in-the-money.

The value of C_2 can also be expressed as a present value formula. This time, however, the value is the sum of two components—(a) the present value of the expected early exercise value of the call half-way through the option's life conditional upon the call being exercised early times the probability that the call will be exercised early, and (b) the present value of the expected terminal value of the call conditional upon the call not being exercised early and being in-the-money at expiration times the probability that the call is not exercised early and is in-the-money at expiration.

The obvious question to ask at this point is, what determines whether the option will be exercised early at time t? The answer lies in considering the call option holder's dilemma at the early exercise opportunity at time t. Figure B illustrates this. Just prior to the early exercise instant, the exercisable proceeds of the call are $F_t - X$. If the call option holder chooses not to exercise, he is left with a European call option with a value of $c(F_t,T - t;X)$. The critical futures price F_t^* is determined by the intersection of $F_t - X$ and $c(F_t,T - t;X)$, or:

$$F_t^* - X = c(F_t^*,T - t;X). \qquad (5)$$

At this point, the option holder is indifferent about early exercise of his option. If the futures price at time t, F_t, is below F_t^*, the option is worth more alive than dead and will be held to expiration. If F_t is above F_t^*, the option is worth more dead than alive and will be exercised early. Note that, given Equation (3), the value of F_t^* may be computed, although a numerical search procedure is required.[20]

With the value of F_t^* determined, the call option pricing formula may be solved. The value of C_2 is as follows:

$$\begin{aligned} C_2 = {} & e^{-rt}[FN_1(a_1) - XN_1(a_2)] \\ & + e^{-rT}[FN_2(-a_1,b_1; -\sqrt{1/2}) \\ & - XN_2(-a_2,b_2; -\sqrt{1/2})], \end{aligned} \qquad (6)$$

where

$$a_1 = [\ln(F/F_t^*) + 0.5\sigma^2 t]/\sigma\sqrt{t},$$

$$a_2 = a_1 - \sigma\sqrt{t},$$

$$b_1 = [\ln(F/X) + 0.5\sigma^2 T]/\sigma\sqrt{T},$$

$$b_2 = b_1 - \sigma\sqrt{T}, \text{ and } t = T/2.$$

$N_1(a)$ represents the probability that a unit normally distributed variable x will be less than or

Table V Valuation of an American Call Option on A Futures Contract Using the Compound Option Valuation Approach (X = 100, r = 0.12, σ = 0.20, T = 0.25)

Futures Price (F)	American Call Option Price Sequence $C_1(F,T;X)$	$C_2(F,T;X)$	$C_3(F,T;X)$	Analytic Approx. $C(F,T;X)_a$	Numerical Approx. $C(F,T;X)_n$
80	0.04	0.04	0.04	0.04	0.04
90	0.69	0.69	0.69	0.69	0.69
100	3.87	3.88	3.88	3.90	3.89
110	10.63	10.71	10.73	10.76	10.76
120	19.55	19.80	19.88	20.02	20.01

Table VI Theoretical European and American Futures Option Values (exercise price (X) = 100)

Option Parameters	Futures Price (F)	Call Options: European $c(F,T;X)$	Call Options: American $C(F,T;X)_a$	Call Options: American $C(F,T;X)_n$	Put Options: European $p(F,T;X)$	Put Options: American $P(F,T;X)_a$	Put Options: American $P(F,T;X)_n$
r = 0.12	80	0.04	0.04	0.04	19.45	19.99	20.00
σ = 0.20	90	0.69	0.69	0.69	10.40	10.53	10.53
T = 0.25	100	3.87	3.90	3.89	3.87	3.90	3.89
	110	10.63	10.76	10.76	0.93	0.93	0.93
	120	19.55	20.02	20.01	0.14	0.14	0.14
r = 0.16	80	0.04	0.04	0.04	19.25	19.99	20.00
σ = 0.20	90	0.68	0.69	0.69	10.29	10.49	10.48
T = 0.25	100	3.83	3.87	3.86	3.83	3.87	3.86
	110	10.52	10.70	10.71	0.92	0.92	0.92
	120	19.36	20.01	20.00	0.14	0.14	0.14
r = 0.12	80	1.15	1.15	1.15	20.56	20.84	20.84
σ = 0.40	90	3.48	3.50	3.49	13.19	13.30	13.30
T = 0.25	100	7.73	7.78	7.77	7.73	7.78	7.77
	110	13.87	13.98	13.98	4.17	4.19	4.18
	120	21.49	21.74	21.74	2.08	2.09	2.08
r = 0.12	80	0.29	0.29	0.29	19.13	20.04	20.01
σ = 0.20	90	1.67	1.68	1.68	11.09	11.35	11.36
T = 0.50	100	5.31	5.38	5.38	5.31	5.38	5.38
	110	11.50	11.76	11.77	2.08	2.10	2.10
	120	19.51	20.28	20.24	0.68	0.68	0.68

equal to a, and $N_2(a,b;\rho)$ is the probability that two unit normally distributed random variables x and y with correlation ρ will be less than or equal to a and b, respectively.[21] The first term in brackets in Equation (6) is the expected call option value at the early exercise instant conditional upon early exercise times the probability that the call is exercised early. The second bracketed term is the expected call option value at expiration conditional upon the call not being exercised early and being in-the-money at expiration times the probability that the call is not exercised early and is in-the-money at expiration. $N_1(a_2)$ is the probability that the call will be exercised early, and $N_2(-a_2,b_2;-\sqrt{1/2})$ represents the probability that the call will not be exercised early and will be in-the-money at expiration. The formula for C_3 is derived in a similar fashion.

Results

Table V presents call option values for C_1, C_2 and C_3, as well as the approximation value from Equation (4), denoted as $C(F,T;X)_a$. Note that the call with one early exercise opportunity has a greater value than the call with no early exercise, and the call with two early exercise opportunities has a greater value than the call with one early exercise opportunity. Each additional early exercise opportunity adds value to the option.

Table V also includes a column labelled $C(F,T;X)_n$. These American futures option values were computed using numerical methods. These methods, while accurate in the sense that they account for the "infinite" number of early exercise opportunities offered the American option holder, are computationally expensive and are not sensible for microcomputer applications.[22] It is encouraging, however, that the extrapolated values $C(F,T;X)_a$ very closely match the numerically estimated values. Furthermore, they do so at *less than 5 per cent of the computational cost* and can be easily programmed on a microcomputer!

Table VI presents a sensitivity analysis of the theoretical European and American futures option values for a variety of option pricing parameters. It is important to see how well the extrapolation method works for reasonable ranges of inputs in the option pricing formula. Although the valuation equations for put option contracts are not presented here, their specifications are easily derived from the call option pricing results provided, and their values are included in Table VI.

Note that the extrapolation method yields option prices within a penny or so of the numerical method. Assuming the numerical method provides the "true" value of the American futures option (and considering the numerical method's computational cost), this result is impressive. Note also that the degree of mispricing is greater, the further the option is in-the-money and the longer is its time to expiration. Even under these circumstances, however, the relative error is less than two-tenths of 1 per cent. A final observation is that out-of-the-money options generally have negligible early exercise premiums. This suggests that it may be appropriate to apply the computationally less expensive Black model, Equation (3), to approximate the values of out-of-the-money American futures options.

It should be emphasized that the results presented here apply to all futures option contracts, independent of the nature of the commodities underlying the futures. Futures options written on financial instrument, foreign currency, precious metal and agricultural futures contracts all follow the valuation principles discussed. Such general results are not available, however, for options written on the spot commodities themselves. ■

Footnotes

1. Although not futures options *per se*, commodity options were introduced in the U.S. as early as the mid-1800s. The Commodity Exchange Act of 1936, however, banned trading in options on all domestic, regulated commodites, and it was not until 1982 that the commodity futures options appeared on the scene. For some early perspectives on agricultural options trading, see P. Mehl, *Trading in Privileges on the Chicago Board of Trade* (Washington, D.C.: United States Department of Agriculture, Circular No. 323, December 1934).
2. See F. Black, "The Pricing of Commodity Contracts," *Journal of Financial Economics* 3 (1976), pp. 167–179.
3. See, for example, E. Moriarty, S. Phillips and P. Tosini, "A Comparison of Options and Futures in the Management of Portfolio Risk," *Financial Analysts Journal*, January/February 1981, pp 61–67; M. R. Asay, "A Note on the Design of Commodity Contracts," *Journal of Futures Markets* 2 (1982), pp. 1–7; and A. Wolf, "Fundamentals of Commodity Options on Futures," *Journal of Futures Markets* (1982), pp 391–408.
4. See R. E. Whaley, "Valuation of American Futures Options: Theory and Empirical Tests," *Journal of Finance*, March 1986.
5. J. C. Cox, J. E. Ingersoll and S. A. Ross, "The Relation Between Forward and Futures Prices," *Journal of Financial Economics* 9 (1982), pp. 321–346, demonstrate that the price of a futures contract is equal to the price of a forward contract when interest rates are nonstochastic.
6. The riskless bonds used throughout the study may be thought of as Treasury bills with time T remaining to maturity.
7. The concept of a "rollover" futures strategy was introduced by Cox, Ingersoll and Ross, "The Relation Between Forward and Futures Prices," *op. cit.*
8. To understand the nature of the rollover strategy, consider the first day's settlement activity. At the beginning of the first day, $e^{-r(T-1)}$ futures contracts are purchased at price F_0. At the end of the day, the contracts are marked to market, and the long registers a gain (loss) of $e^{-r(T-1)}(F_1 - F_0)$. On the second day, the gain is $e^{-r(T-2)}(F_2 - F_1)$, and on the third day, $e^{-r(T-3)}(F_3 - F_2)$. If each of these daily gains (losses) is then invested at the riskless rate until the end of the futures contract life, the terminal value of the rollover futures position will be as follows:

$$\begin{aligned} & e^{-r(T-1)}(F_1 - F_0)e^{r(T-1)} + e^{-r(T-2)}(F_2 - F_1)e^{r(T-2)} \\ & + e^{-r(T-3)}(F_3 - F_2)e^{r(T-3)} + \dots \\ & + e^{-r(T-t)}(F_t - F_{t-1})e^{r(T-t)} + \dots \\ & + (F_T - F_{T-1}) = F_T - F_0 = S_T - F_0. \end{aligned}$$

Note that it is this long rollover futures position that provides the same terminal value as a single long forward contract.

9. Conversion and reverse conversion trading strategies are explained in detail in L.G. McMillan, *Options as a Strategic Investment: A Comprehensive Analysis of Listed Stock Option Strategies* (New York: New York Institute of Finance, 1980), pp. 238–240.
10. R.C. Merton, "The Theory of Rational Option Pricing," *Bell Journal of Economics and Management Science* 4 (1973), pp. 141–183, provides a comprehensive analysis of the rational stock option pricing relations.
11. In H.R. Stoll, "The Relationship Between Put and Call Option Prices," *Journal of Finance* 24 (1969), pp. 802–824, put-call parity for European stock options was shown to be:

$$c(S,T;X) - p(S,T;X) = S - Xe^{-rT}$$

The structure of this put-call parity condition is the same as that demonstrated for European futures options [i.e., Equation (1)], if the futures, futures options and the stock options expire at the same instant and if the cost-of-carry relation $F = Se^{rT}$ holds.

12. See Moriarty, Phillips and Tosini, "A Comparison of Options and Futures," *op. cit.* and Wolf, "Fundamentals of Commodity Options on Futures," *op. cit.*
13. This relation first appeared in H.R. Stoll and R.E. Whaley, "New Option Instruments: Arbitrageable Linkages and Valuation," *Advances in Futures and Option Research* 1 (forthcoming, 1986). A partial result appears in K. Ramaswamy and S. Sundaresan, "The Valuation of Options on Futures Contracts," *Journal of Finance*, December 1985.
14. It is worthwhile to note that the Black model is not unlike the Black-Scholes model for pricing the European call option on a non-dividend-paying stock. (See F. Black and M. Scholes, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy* 81 (1973), pp. 637–659.) If the cost-of-carry relation $F = Se^{rT}$ is substituted into Equation (3), one obtains the Black-Scholes model:

$$c(S,T;X) = SN(d_1) - Xe^{-rT}N(d_2)$$

where $d_1 = [\ln(S/X) + (r + 0.5\sigma^2)T]/\sigma\sqrt{T}$. This result was first noted by F. Black, "The Pricing of Commodity Contracts," *op. cit.* and then later by M.R. Asay, "A Note on the Design of Commodity Contracts," *Journal of Futures Markets* 2 (1982), pp. 1–7.

15. This interpretation of the Black model invokes the risk-neutrality argument that appears in J.C. Cox and S.A. Ross, "The Valuation of Options for Alternative Stochastic Processes," *Journal of Financial Economics* 3 (1976), pp. 145–166. If a riskless hedge may be formed between the futures option and the underlying futures contract, the value of the option is the same for risk-neutral investors and for risk-averse investors. Thus, for mathematical tractability, assume investors are risk-neutral. The appropriate discount rate to use in the present value computation is, therefore, the riskless rate of interest.
16. The pricing equation for a European put option on a futures contract may be derived by substituting the European call option valuation Equation (3) into the put-call parity Equation (1). The value of the European put option is as follows:

$$p(F,T;X) = e^{-rT}[XN(-d_2) - FN(-d_1)],$$

where d_1 and d_2 are the same as they were defined for Equation (3). Here, the term in brackets is the expected value of the put option at expiration conditional upon the option being in-the-money at expiration times the probability that the put option will finish in-the-money; $N(-d_2)$ is the probability that the futures price will be below the exercise price at expiration. Note that when the European call and put options have the same exercise price and time to expiration, the probability that the call will finish in-the-money, $N(d_2)$, and the probability that the put will finish in-the-money, $N(-d_2)$, sum to one.

17. As Merton ("The Theory of Rational Option Pricing," *op. cit.*) demonstrates, because the exercisable value of an American call option on a non-dividend-paying stock, $S - X$, is always less than the lower price bound of the corresponding European option, $S - Xe^{-rT}$, the American call option is always worth more "alive" than "dead" and will thus not be exercised early.
18. The approach used parallels the methodology used by R. Geske and H.E. Johnson, "The American Put Valued Analytically," *Journal of Finance* 39 (1984), pp. 1511–1524, in deriving the analytical formula for an American put option on a stock. The compound option valuation approach was also used to price the American call option on a dividend-paying-stock. See R. Roll, "An Analytic Valuation Formula for Unprotected American Call Options on Stocks with Known Dividends," *Journal of Financial Economics* 5 (1977), pp. 251–258; R. Geske, "A Note on an Analytical Formula for Unprotected American Call Options on Stocks with Known Dividends," *Journal of Financial Economics* 7 (1979), pp. 375–380; and R.E. Whaley, "On the Valuation of American Call Options on Stocks with Known Dividends," *Journal of Financial Economics* 9 (1981), pp. 207–211.
19. The derivation of these weights is provided in R. Geske and H.E. Johnson, "The American Put Valued Analytically," *op. cit.*

20. As a numerical example, consider a futures option with an exercise price (X) of 100 and a time to expiration (T) of 0.25. Also, suppose that the riskless rate of interest (r) is 12 per cent and that the standard deviation of the relative price changes in the futures contract (σ) is 20 per cent. The critical futures price F_t^* above which the call option holder will exercise his option at the early exercise opportunity t is 111.84.
21. Methods for evaluating the probabilities $N_1(.)$ and $N_2(.)$ are available in M. Abramowitz and I. Stegum, *Handbook of Mathematical Functions* (Washington, D.C.: National Bureau of Standards).
22. A variety of numerical methods have been applied to American option pricing problems. An interested reader may refer to M. J. Brennan and E.S. Schwartz, "Finite Difference Methods and Jump Processes Arising in the Pricing of Contingent Claims: A Synthesis," *Journal of Financial and Quantitative Analysis* 13 (1978), pp. 461–474, and R. Geske and K. Shastri, "Valuation by Approximation: A Comparison of Alternative Valuation Techniques," *Journal of Financial and Quantitative Analysis* 20 (1985).